FREE CHOICE IN MATHEMATICAL LANGUAGE

ABSTRACT. Breaking with traditional antipragmatic assumptions, recent discussion in this journal submits evidence for pragmatic inferences in mathematical language. The observation is that the interpretation of crucial expressions, e.g., positive quantifiers, results from a truth-conditional enrichment of their literal meaning. While providing general insights into mathematical language, the authors unfortunately do not provide a clear and testable linguistic analysis of the observed inference. However, in this contribution we provide support to their informal proposal by presenting a formal reconstruction of the observed inference as a free choice implicature. In particular, we submit an algorithm based on recursive exhaustification limited to domain alternatives that generates the desired reading under a natural assumption of homogeneity. Our argument is also based on independent and, to our knowledge, hitherto unnoticed evidence for free choice interpretations in mathematics.

1. INTRODUCTION: PRAGMATICS AND MATHEMATICAL LANGUAGE

While most mathematical statements can be formalized, e.g. in first- or second-order logic, the use of natural language in mathematics, "*even in discussions of its foundation*" (Kamp 2020, p. 94), remains central and widespead. In fact, mathematical papers, as they appear in standard mathematical journals, are always written in some natural language extended with a finite collection of formal symbols; call *mathematical language* this mixed language in which standard mathematical papers are written, crucially including definitions, proofs and theorems. *Prima facie*, the use of natural language in mathematics might seem a potential source of confusion: how could one assume the multiple ambiguities which are essential to natural languages, e.g. in meaning derivation, and meet the standards of clarity and precision that mathematical reasoning most certainly require?

In order to escape confusion, one may be tempted to radically obliterate natural language ambiguities from mathematics so as to conceive mathematical language as a kind of heuristic, useful in general but pretty problematic if taken too seriously. In particular, one may assume that meaning in mathematical language is always equivalent to the literal meaning of expressions, e.g. that the meaning of a natural language quantifier amounts to the meaning of its logical counterpart. The latter thesis, which builds on a long implicit tradition in philosophy, linguistics, and mathematics (see discussion in Ruffino et al. 2020b for details), has been explicity defended for instance in Ganesalingam (2013).

(1) **Antipragmatic Thesis (APT):** Meaning in mathematical language is equivalent to the literal meaning of expressions.

In a recent contribution appeared in this journal, Ruffino et al. (2020a) directly oppose to the strategy we just mentioned by submitting examples such as (2a), crucially involving the interpretation of the existential quantifier as used in a theorem from computability theory. Their observation is that in the context of computability theory the natural interpretation of the existential quantifier gives rise to enriched contents such as (2b).

Key words and phrases. Pragmatics and mathematical language, Free choice implicatures, Existential quantifier, Computability.

(2) a. *There is one computable subset of a given computably enumerable set.*

b. → *There are infinitely many computable subsets of a given computably enumerable set.*

Since the enriched content observed is not entailed by the literal meaning of the existential quantifier, or by the literal meaning of the other expressions contained in (2a), their conclusion is that "*pragmatics phenomena arise even at the most fundamental level of mathematical discourse*" (Ruffino et al. 2020a, p. 116), in clear contradiction with (1). Note that, if their argument is solid and can be generalized, a different interpretation concerning the use of natural language in mathematics becomes immediately available: indeed, one may not only realize that no confusion is actually engendered, but claim in addition that the very possibility of drawing pragmatic inferences is what motivates such use. Even in mathematics, we don't use of natural language *despite* its multiple ambiguities, but precisely *on account of* the rich complexity it encompasses.

Unfortunately, while offering general insights into mathematical language, Ruffino et al. do not provide a clear and testable linguistic account of how the inference to (2b) is derived from the sentence (2a). Our aim in this contribution is to link their observation to similar phenomena attested in natural language interpretation. In particular, assuming a standard semantics for disjunction and for the existential, we concur with the authors that the observed inference cannot be an entailment from the literal meaning, whence the need for an additional, enriching interpretive mechanism. Our specific proposal is that such an additional mechanism amounts to a *free choice* implicature. This is a phenomenon observed early in the study of natural language meaning (Von Wright 1968; Kamp 1974) and recently at the center of an intense debate in the linguistic literature (see Meyer 2015 for details). We submit in particular a formal reconstruction of the observed inference as a free choice implicature obtained via recursive exhaustification limited to domain alternatives. Our argument is also based on independent and, to our knowledge, hitherto unnoticed evidence for free choice interpretations in mathematics. In conclusion, our understanding is that this reconstruction provides support to Ruffino et al.'s informal proposal that mathematical language involves pragmatic procedures while also advancing the linguistic debate concerning the free choice mechanisms.

2. FREE CHOICE IN NATURAL LANGUAGE

Let us begin by briefly illustrating the mechanism of free choice in natural language. Consider the sentences in (3a) and (4a). In these examples, a standard modal and an existential quantifier are respectively shown to range over a disjunctive structure based on a finite domain. The puzzling observation with respect to these examples is that they are associated with the conjunctive interpretations observed in (3b) and (4b), apparently conflicting with a standard semantics for modality and quantification and on a Boolean analysis of disjunction. In (3a-i), (4a-i), and (3b-i), (4b-i) we submit a formal reconstruction of sentences and interpretations.

(3) a. *You may have an apple or a pear.*

i. $\Diamond(\mathbf{a} \vee \mathbf{b})$

b. → *You may have an apple and you may have a pear.*

i. $\Diamond\, \mathbf{a} \wedge \Diamond\, \mathbf{b}$

(4) a. *There is beer in the fridge or in the ice-bucket.*

i. $\exists x((\mathbf{B}x \wedge \mathbf{F}x) \vee (\mathbf{B}x \wedge \mathbf{I}x))$

b. → *There is beer in the fridge and there is beer in the ice-bucket.*

i. $\exists x(\mathbf{B}x \wedge \mathbf{F}x) \wedge \exists x(\mathbf{B}x \wedge \mathbf{I}x)$

One prominent approach in the linguistic literature derives the observed interpretations via free choice implicature (see Fox 2007; Klinedinst 2007; Santorio & Romoli 2017; Bar-Lev 2018 among others). Although individual theories differ in technical details, given our current purposes here we focus on three assumptions of this implicature approach. To begin with, we assume that free choice implicatures are derived via application of the exhaustivity operator '$\mathbf{O}$', an implicit counterpart of the exclusive particle *only*, negating a suitably derived set of alternatives. Note that, in recent literature on implicature, the exhaustivity operator is taken to be also responsible for the derivation of standard scalar implicatures (see Chierchia et al. 2012; Sauerland & Stateva 2007; Pistoia-Reda 2014 among many others). In addition, we assume that the operator applies recursively; more precisely, we assume that the operator ranges over alternatives that, being intended as potential utterances, are already exhaustified (see Fox 2007 for details). Finally, we assume that the set of negatable alternatives crucially includes domain alternatives (in the sense of Sauerland 2004); in particular, the set of negatable alternatives associated with (3a) and (4a) above is assumed to contain exhaustifications of the single disjuncts (i.e. of '$\Diamond\mathbf{a}$' and '$\Diamond\mathbf{b}$', and of '$\exists x(\mathbf{B}x \wedge \mathbf{F}x)$' and '$\exists x(\mathbf{B}x \wedge \mathbf{I}x)$') in addition to exhaustifications of the standard scalar conjunctive alternatives (i.e. of '$\Diamond(\mathbf{a} \wedge \mathbf{b})$', and of '$\exists x((\mathbf{B}x \wedge \mathbf{F}x) \wedge (\mathbf{B}x \wedge \mathbf{I}x))$'). As it is easy to show, the combination of these assumptions results in the algorithms observed in (A) and in (B), which in turn are easily shown to derive the desired interpretations. In (B), for simplicity we denote '$[\mathbf{B}x \wedge \mathbf{F}x]$' by '$\mathbf{B}_0 x$' and '$\mathbf{B} \wedge \mathbf{I}x$' by '$\mathbf{B}_1 x$'.

$$(A) \qquad \underbrace{\mathbf{O}\,[\mathbf{O}\,[\Diamond(\mathbf{a} \vee \mathbf{b})]]}_{\alpha_0} = \underbrace{\Diamond(\mathbf{a} \vee \mathbf{b}))\ \wedge\ \neg\mathbf{O}[\Diamond\mathbf{a}]\ \wedge\ \neg\mathbf{O}[\Diamond\mathbf{b}]}_{\beta_0} = \underbrace{\Diamond\mathbf{a} \wedge \Diamond\mathbf{b}}_{\gamma_0}$$

$$(B) \qquad \underbrace{\mathbf{O}\,[\mathbf{O}\,[\exists x(\mathbf{B}_0 x \vee \mathbf{B}_1 x)]]}_{\alpha_1} = \underbrace{\exists x(\mathbf{B}_0 x \vee \mathbf{B}_1 x)\ \wedge\ \neg\mathbf{O}[\exists x\mathbf{B}_0 x]\ \wedge\ \neg\mathbf{O}[\exists x\mathbf{B}_1 x]}_{\beta_1} = \underbrace{\exists x\mathbf{B}_0 x \wedge \exists x\mathbf{B}_1 x}_{\gamma_1}$$

Remark 2.1. *Intuitively, β_1 means that $\mathbf{B}_0$ or $\mathbf{B}_1$ are satisfied, but it cannot be the case that only of the two is satisfied, from which it immediately follows that both $\mathbf{B_0}$ and $\mathbf{B_1}$ are satisfied (and this coincides with γ_1); β_0, and its relation to γ_0, can be similarly understood.*

Our claim in this paper is that the phenomenon observed by Ruffino et al. (i.e. the inference going from (2a), a disjunction over an infinite domain stating that there exists at least one computable subset of a given computably enumerable set, to (2b), a conjunction over the same infinite domain stating that there are infinetely many such computable subsets) can be derived as a free choice implicature in mathematical language. This, according to us, provides evidence for pragmatic procedures in mathematics.

The proposal is organized in two steps. First, in §3.1 we prove that (2a) is formally equivalent to (4a), under a natural principle of homogeneity to be clarified; then, in §3.2 we show that (2b) can be derived from (2a) via recursive exhaustification of domain alternatives. The latter restriction is plausibly entailed by the principle discussed.

Before we proceed, let us conclude this section by submitting the crucial observation that free choice interpretations are quite common in mathematical language. For illustration, note that intuitively the examples given below in (5), which might easily occur in mathematical papers, are formally similar to sentence (3a) above, showing a standard modal ranging over a disjunction, and their interpretations enjoy the same conjunctive flavor as the one observed in (3b). This observation provides indirect evidence in favor of our point. Against this background, we provide evidence for free choice implicatures being observed in mathematical language also under the existential configuration observed in (4a).

(5) "these graphs can be connected or disconnected", "x can be even or odd", "the cardinality of Y can be finite or countable", "u and v can be either 0 or 1".

3. Free choice in mathematical language

3.1. **From the existential quantifier to the infinite disjunction.** Recall that a set A of non-negative integers is *computably enumerable* if A is the range of a computable function f (i.e., if there is a Turing machine which lists, possibly with repetitions, all and only the elements of A). It is not hard to see that, if such an f is strictly increasing, then A is also computable: indeed, to check whether a given input x is in A, it is enough to find the least y such that $f(y) \geq x$ and see if $f(y) = x$. So, in order to prove that (2a) holds, it suffices to prove the following claim,

(6) Any computably enumerable set A contains the range of a computable strictly increasing function.

A standard way to prove (6) is as follows: One fixes a computable function g which computably enumerates A and defines a computable strictly increasing function h with $\mathrm{range}(h) \subseteq A$ by setting

$$(C) \qquad \begin{cases} h(0) = g(0); \\ h(x+1) = g(\min\{y : g(y) > h(x)\}). \end{cases}$$

That is, h skips the outputs of g that are smaller than previous ones. The desired computable subset of A is then $\mathrm{range}(h)$.

Now, observe that a function which satisfies (6) could have defined in a different way, e.g., in the inductive of step of (C), instead of taking the least y such that $g(y)$ is bigger than all previous outputs, we could have taken the second least number satisfying this property. By such a choice, we would have constructed a different computable function, giving rise to a different computable subset of A. More generally, there are infinitely many ways of choosing elements from the range of g that generate a computable subset of A. As long as the function is kept computable and strictly increasing, these choices are irrelevant to the proof of (6).

The last remark reflects a certain *principle of homogeneity*, denoted by $\mathbf{H}$, which states that most computational properties considered in computability theory are invariant with respect to computable modifications[1]. In particular, we have that any finite modification of a computable set is still computable. We submit that $\mathbf{H}$ is the device that allows the inferential step from (2a) to (2b). To see this, let us reconsider the proof of (2a) through the lens of $\mathbf{H}$. Now, instead of constructing a single function h, we acknowledge that, for any input i, there are multiple outputs which keep the desired function computable and

[1] More formally, they are preserved under computable isomorphisms.

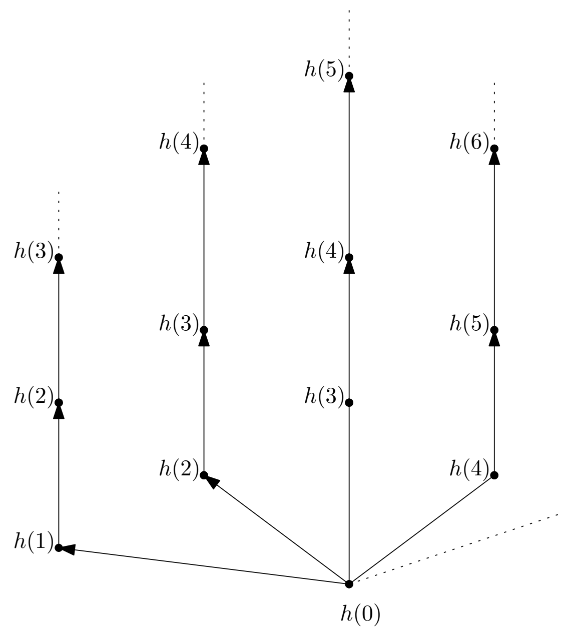

FIGURE 1. A finite fragment of $\mathbf{T}$

stricly increasing. One could visualize these choices into a labelled tree $\mathbf{T}$, where each node is labelled by one of the output candidates for our function on a given input i, and each path corresponds to a computable subset of A. We denote our labelling system as l.

For the sake of simplicity, let us construct $\mathbf{T}$ as follows (see Figure 1):

(1) The root ρ of $\mathbf{T}$ is labelled by $h(0)$, i.e., $l(\rho) := h(0)$;
(2) ρ is the only branching node of $\mathbf{T}$ and it has infinitely many successors, which are respectively labelled by all the $h(i)$'s for $i > 0$;
(3) each node ν with depth > 1 has a unique successor μ, and $l(\mu) := h(h^{-1}l(\nu) + 1)$.

It is not hard to see that the labels lying on any path through $\mathbf{T}$ naturally give rise to a computable subset of A. Indeed, the leftmost path of $\mathbf{T}$ corresponds to $\mathrm{range}(h)$, and the following hold:

a) any other path is strictly contained in $\mathrm{range}(h)$;
b) any two paths differ only finitely.

Hence, it follows from $\mathbf{H}$ that the paths through to $\mathbf{T}$ correspond to an infinite collection of a computable subsets of A. So, relatively to $\mathbf{T}$, one can rephrase (2) as the following infinite disjunction,

(7) *There is* a computable strictly increasing function j_0 such that $\mathrm{range}(j_0) \subseteq A$ and $\mathrm{range}(j_0) = \mathrm{range}(h)$ *or* a computable strictly increasing function j_1 such that $\mathrm{range}(j_1) \subseteq A$ and $\mathrm{range}(j_1) = \mathrm{range}(h) \setminus \{h(1)\}$ *or* ... *or* a computable strictly increasing function j_i such that $\mathrm{range}(j_i) \subseteq A$ and $\mathrm{range}(j_i) = \mathrm{range}(h) \setminus \{h(n) : 0 < n \leq i\}$ *or* ...

Denote by $\mathbf{T}$ the second-order predicates $\mathbf{C}x, \mathbf{I}x, \mathbf{R}_i x$ which respectively mean "x is a computable function", "x is strictly increasing", "the range of x does not overlap the set

$\{h(n) : 0 < n \leq i\}$". Then, (7) can be formalized as

$$(D) \qquad \exists x[\bigvee_{i\in\mathbb{N}}(\mathbf{C}x \wedge \mathbf{I}x \wedge \mathbf{R}_i x)].$$

3.2. **From the infinite disjunction to the infinite conjuction.** Once we have the formal equivalence between (2a) and (4a) (via (7) and under the natural assumption of homogeneity clarified), it is easy to realize that the inference observed by Ruffino et al. can be described as a free choice implicature. In particular, the implicature, obtained through a recursive exhaustification mechanism, gives rise to a conjunctive interpretation, which is easily shown to correspond to (2b) above. Note that the derived interpretation clearly parallels the one derived as part of the natural language mechanism, i.e. (4b), with the only exception of the mechanism now ranging over an infinite domain.

As a special assumption of our account, note that the set of negatable alternatives in our case is crucially forced to only include domain alternatives; in other words, we observe that the scalar, conjunctive alternative must be excluded from the set in order for the desired interpretation to be derived. Consequently, our account involves exhaustifications of the single disjuncts (i.e. of '$\exists x$ $[\mathbf{C}x \wedge \mathbf{I}x \wedge \mathbf{R}_i x]$' with $i \in \mathbb{N}$) while ignoring, in difference from the natural language mechanism, exhaustifications of the conjunctive alternative (i.e. of '$\exists x[\bigwedge_{i\in\mathbb{N}}(\mathbf{C}x\wedge\mathbf{I}x\wedge\mathbf{R}_i x)]$'). Note that, given homogeneity, negating the conjunctive alternative appears to be incompatible with the base disjunction; as a consequence, ignorance of the conjunctive alternative may be motivated based on a standard assumption of non-contradictoriness of the negatable alternatives. Thus, we conjencture that standard scalar implicatures, crucially involving conjunctive alternatives, could be observed in mathematical language in contexts without homogenity. The algorithm below is clearly shown to derive the desired interpretation. To simplify our discussion, let us denote by '$\mathbf{P}_i x$' the formula '$\mathbf{C}x \wedge \mathbf{I}x \wedge \mathbf{R}_i x$'; hence, in particular, we take '$\mathbf{P}_i x$' to mean "x is a computable strictly increasing function whose range does not not overlap the set $\{h(n) : 0 < n \leq i\}$".

$$(E) \qquad \underbrace{\mathbf{O}\left[\mathbf{O}\left[\exists x \bigvee_{i\in\mathbb{N}} \mathbf{P}_i x\right]\right]}_{(\alpha_2)} = \underbrace{\left(\exists x \bigvee_{i\in\mathbb{N}} \mathbf{P}_i x\right) \wedge \left(\bigwedge_{I\subset\mathbb{N}} \neg\mathbf{O}\left[\exists x \bigvee_{i\in I} \mathbf{P}_i x\right]\right)}_{(\beta_2)} = \underbrace{\bigwedge_{i\in\mathbb{N}} \exists x \mathbf{P}_i x}_{(\gamma_2)}$$

The above equation is crucially a generalization of (B). The additional complexity comes from the fact that the algorithm now deals with an infinite domain. In particular, in (B) the domain alternatives to be negated by the recursive exhaustification operator are only two; here the right-conjuct of (β) prevents the exhaustification of infinitely many domain alternatives, corresponding to all proper subcollection of $\mathbf{P}_i$-formulas. However, (E) is a natural generalization of (B).

Remark 3.1. *In analogy with Remark 2.1, let us conclude this section by offering to the interested reader the proof that (β_2) implies (γ_2). Towards a contradiction, suppose that (β_2) is true and (γ_2) is false. In particular, denote by T the set of numbers such that $\mathbf{P}_i$ holds (note that $T \neq \mathbb{N}$, otherwise γ would be true). Now, observe that, for all $S \supset T$, the following holds*

$$\left(\bigvee_{i\in T} \mathbf{P}_i x\right) \wedge \left(\neg \bigvee_{i\in S} \mathbf{P}_i x\right).$$

Hence, the collection of $\mathbf{P}_i$-formulas indexed by T is exhaustified, and therefore the following holds

$$\neg\mathbf{O}\left[\bigvee_{i\in I}\mathbf{P}_i x\right].$$

But this makes (β_2) false, contradicting our hypothesis. So, (β_2) implies (γ_2); all other implications should be easy to prove.

4. Conclusion

In this contribution we discussed the possibility that a suitably defined mathematical language includes pragmatic procedures. Our proposal builds on a recent article appeared in this journal (Ruffino et al. 2020a), in which traditional antipragmatic assumptions are criticized based on the observation that certain crucial expressions, e.g. quantifiers, are sometimes interpreted within mathematical language under truth-conditional enrichments of their literal meaning. In this contribution we presented a formal reconstruction of the observed inference as a *free choice* implicature. In particular, we described a mechanism based on recursive exhaustification limited to domain alternatives under a natural assumption of homogeneity. Our contribution then provides evidence in favor of pragmatic inferences in mathematics, by submitting an explicit and testable algorithm for the derivation, and also contributes new observation for the linguistic discussion concerning the nature of the *free choice* mechanisms.

References

Bar-Lev, Moshe E. 2018. *Free choice, homogeneity, and innocent inclusion*: Hebrew University of Jerusalem dissertation.

Chierchia, Gennaro, Danny Fox & Benjamin Spector. 2012. Scalar implicature as a grammatical phenomenon. In C. Maienborn, von Heusinger K. & Portner P. (eds.), *Semantics: An International Handbook of Natural Language Meaning*, vol. 3, 2297–2331. Berlin: Mouton de Gruyter.

Fox, Danny. 2007. Free choice and the theory of scalar implicatures. In Sauerland & Stateva (2007) 71–120.

Ganesalingam, Mohan. 2013. The language of mathematics. In *The Language of Mathematics*, 17–38. Springer.

Kamp, Hans. 1974. Free choice permission. *Proceedings of the Aristotelian Society* 74. 57–74.

Kamp, Hans. 2020. Trying to see and say where I am, have been and would want to go.

Klinedinst, Nathan Winter. 2007. *Plurality and possibility*: UCLA dissertation.

Meyer, Marie-Christine. 2015. Generalized free choice and missing alternatives. *Journal of Semantics* 33(4). 703–754.

Pistoia-Reda, Salvatore (ed.). 2014. *Pragmatics, semantics and the case of scalar implicatures* Palgrave studies in pragmatics, language and cognition. London: Palgrave Macmillan.

Ruffino, Marco, Luca San Mauro & Giorgio Venturi. 2020a. At least one black sheep: Pragmatics and mathematical language. *Journal of Pragmatics* 160. 114 – 119.

Ruffino, Marco, Luca San Mauro & Giorgio Venturi. 2020b. Speech acts in mathematics. *Synthese* 1–25.

Santorio, Paolo & Jacopo Romoli. 2017. Probability and implicatures: A unified account of the scalar effects of disjunction under modals. *Semantics and Pragmatics* 10. 13.

Sauerland, Uli. 2004. Scalar implicatures in complex sentences. *Linguistics and philosophy* 27(3). 367–391.

Sauerland, Uli & Penka Stateva (eds.). 2007. *Presupposition and implicature in compositional semantics* Palgrave studies in pragmatics, language and cognition. London: Palgrave Macmillan.

Von Wright, Georg Henrik. 1968. *An essay in deontic logic and the general theory of action*. Amsterdam: North-Holland Publishing Company.