Predicativity and the Russell-Myhill Paradox of Propositions: A Contribution to the Development of Church's Intensional Logic

Sean Walsh[*]

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Abstract

 This paper sets out a predicative response to the Russell-Myhill paradox of propositions within the framework of Church's intensional logic. A predicative response places restrictions on the full comprehension schema, which asserts that every formula determines a higher-order entity. In addition to motivating the restriction on the comprehension schema from intuitions about the stability of reference, this paper contains a consistency proof for the predicative response to the Russell-Myhill paradox. The models used to establish this consistency also model other axioms of Church's intensional logic that have been criticized by Parsons and Klement: this, it turns out, is due to resources which also permit an interpretation of a fragment of Gallin's intensional logic. Finally, the relation between the predicative response to the Russell-Myhill paradox of propositions and the Russell paradox of sets is discussed, and it is shown that the predicative conception of set induced by this predicative intensional logic allows one to respond to the Wehmeier problem of many non-extensions.

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[*]Department of Logic and Philosophy of Science, 5100 Social Science Plaza, University of California, Irvine, Irvine, CA 92697-5100, U.S.A., swalsh108@gmail.com or walsh108@uci.edu

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1 Introduction

The Russell-Myhill paradox is a paradox about propositions which is structurally analogous to the Russell paradox about sets. While predicativity has been well-explored as a response to the Russell paradox about sets, it seems that there has been no attempt to set out and analyze a predicative response to the Russell-Myhill paradox. The primary aim of this paper is to do just that. The crucial idea behind the predicativity response is to restrict the comprehension schema for the ambient higher-order logic. Intuitively, the comprehension schema says that every well-formed formula determines a higher-order entity. So besides the burden of showing formal consistency, a predicative response to the Russell-Myhill paradox must provide at least the beginnings of an account of why some but not all formulas succeed in determining higher-order entities.

The resulting formal system whose consistency we establish is centered around the intensional logic of Church. This intensional logic has a neutral core which we exposit in §2; it is neutral in the sense that its axioms are comparatively undemanding and consistent with contemporary theorizing based on possible-world semantics. In the subsequent §3, we set out a formalized version of the Russell-Myhill paradox, which is broadly similar to formalizations offered by Anderson and Klement, and we describe how different models offered by Tichý, Kaplan, and Anderson block different premises in the formalized version of the paradox. Then in §§4-5, we turn to and discuss the predicativity response to the Russell-Myhill paradox. In addition to discussing the philosophical motivations for predicative restrictions, we establish the formal consistency of the system by constructing a series of models.

However, the models offered here validate an additional axiom of Church's intensional logic which, as Parsons and Klement have emphasized, is in some ways contrary to the spirit of a fine-grained theory of propositions. In §6 we explain why this axiom holds on our models: it turns out that this is related to an expressive resource which allows these models to interpret a fragment of Gallin's intensional logic. Finally, in §7, we present one application of a broadly predicative perspective on the Russell-Myhill paradox about propositions, namely a response to the Wehmeier problem of many non-extensions that arises in connection to the naive conception of set found in Frege's *Grundgesetze*.

This paper is the third in a series of papers – the other two being [88], [89]– which collectively constitute a sequel to our paper [87]. In that earlier paper [87], we studied the consistent fragments of Frege's *Grundgesetze* system. In our more recent [88], we used Gödel's constructible sets to study how much of Zermelo-Fraenkel set theory can be consistency realized in these fragments of the *Grundgesetze*. In the complementary paper [89], we showed how a related theory of abstraction principles can deductively effect some of this recovery of fragments of Zermelo-Fraenkel set theory.

However, these papers do not touch the question of whether the models used to prove consistency of the *Grundgesetze* system are anything like intended models. In this paper, we use tools from constructibility in §5 to produce our models of our extension of Church's intensional logic. Our response to the problem of many non-extensions in §7 involves showing how this extension of Church's intensional logic can define a model of Frege's *Grundgesetze* system. So in addition to articulating a predicative response to the Russell-Myhill paradox

of propositions, this paper suggests the possibility of viewing the consistent fragments of Frege's naive set theory through the lens of a consistent version of Church's intensional logic. However, despite these connections to our earlier papers, this paper has been written so that one need not have read these other papers. At one point in §5 below, we reference the earlier paper [88] for examples of one of our defined notions in this paper (namely that of an intensional hierarchy (5.9)). However, this is the only respect in which this paper depends on the earlier papers.

2   The Neutral Core of Church's Intensional Logic

The intensional logic of Church is an attempt to axiomatize Frege's sense-reference distinction. Of course, Frege thought that words not only designate their referent, but also express their sense. Hence, on Frege's view, our words bear two semantic relations to non-linguistic entities, namely they bear the *designation relation* to their referents and they bear the *expression relation* to their senses. So in the crudest of terms, Frege is a semantic dualist. This of course allowed him to say how "the morning star" differs in meaning from the "evening star": while these two linguistic expressions refer to the same planet, they express different senses.

In our view, Church's crucial observation was that semantic dualism induces a canonical non-semantic relation. So the semantic dualist doesn't only think that there are more word-world connections, but they are also committed to an additional *world-world* relation. In the case of Frege, the commitment is to a relationship between the abstract Fregean sense expressed by a linguistic expression and the entity (perhaps a planet) which is denoted by that linguistic expression. This relation is called the *presentation* relation in the literature ([51] p. 155). So one says that sense s *presents* denotation d and one writes $\Delta(s, d)$ precisely in the circumstance where there is a linguistic expression which expresses s and denotes d. The "triangle" notation Δ for the presentation relation is helpful here because it reminds us that a sense s on the bottom-left of the triangle stands in the presentation relation $\Delta(s, d)$ to the denotation d on the bottom-right of the triangle in virtue of its semantic relations to some suppressed linguistic expression standing at the top of the triangle.

So Church proceeded by axiomatizing the presentation relationship. Of course, this is not the only way that one might seek to understand the presentation relationship. With respect to a given formal language, we know now how to recursively define a satisfaction relation in terms of reference, and one might have thought that one ought to proceed similarly with the presentation relationship. However, this procedure would require the notion of sense to be as conceptually transparent as the notion of reference. By contrast, Church's own aim in axiomatizing the presentation relationship was to dissipate outright skepticism about Fregean sense. Here is how Church put the point in a 1943 review of a paper of Quine:

> There remains the important task, which has never been approached, of constructing a formalized semantical system which shall take account of both kinds of meaning, the relation between a name and its denotation, and the relation between a name and its sense. [...] [¶] [...] Ultimately it is only on the basis

of their inclusion in an adequate system of this kind that such otherwise inde-
fensibly vague ideas as 'understanding' of an expression, 'attribute,' 'objectiver
Inhalt des Denkens,' may be regarded as logically significant ([12] p. 47).

So one of the original aims of Church's work was to produce a formal theory whose quantifiers
ranged over Fregean senses and which thus serve to implicitly define the notion of Fregean
sense.

The first component of Church's formal theory concerned the Fregean doctrine that
sense determines reference. So Frege tells us that whenever two linguistic expressions have
the same sense, then if one refers then the other does too and they have the same referent.
When put this way, it automatically suggests the following axiom (Church's Axiom 17, cf.
[14] p. 19, [49] pp. 108-109, [1] p. 220; [2] Axiom C8 p. 377):

(2.1) *Sense Determines Reference*: $(\Delta(s, d_0) \ \& \ \Delta(s, d_1)) \Longrightarrow d_0 = d_1$

Practically, this indicates to us that the presentation relationship is functional in character.
So instead of writing $\Delta(s, d)$, we may write instead $\Delta(s) = d$. Likewise, borrowing notation
from computability theory, sometimes we write $\Delta(s) \downarrow$ to indicate that there is a d such
that $\Delta(s) = d$ (cf. [82] pp. 16-17). Of course, we should keep in mind that on its intended
interpretation, the presentation relation is not a total function. For, any meaningful linguistic
expressions will always have a sense but need not have a referent, and linguistic expressions
which lack a referent will of course not present any referent.

The other of Church's axioms pertain to compositionality. On the side of reference,
Frege postulated a fundamental distinction between objects and concepts. Objects were
the referents of proper names of people and places, whereas concepts were the referents of
predicate-words. In sentences such as "Venus is a planet," we predicate a concept ("being
a planet") of an object ("Venus"). Frege realized that this could be rendered as a case
of functional application, namely $\text{PLANET}(\text{VENUS}) = 1$. In doing this, Frege was able to
combine the compositionality of sentences together with the thesis that sentences refer to
truth-values. So following Frege, we here identify "1" with the truth-value "true" and "0"
with the truth-value "false," and for the sake of simplicity we assume that these are the only
truth-values.

Due to the fact that sense is a determiner of reference, it's natural to think that senses
of sentences are also compositional. Frege called the senses of sentences *thoughts*, so that
sentences express thoughts and refer to truth-values. Just as the reference of a sentence is
a function of the reference of its constituent parts, so Frege and Church hold that the sense
of a sentence (a thought or a proposition) is a function of the senses of its parts. This thus
suggested to Church the following axiom on the presentation relationship (Church's Axiom
15 [14] p. 18, [49] pp. 108-109, [1] p. 219; cf. [2] Axiom C7 p. 377):

(2.2) *Composition Axiom*: $[\Delta(f') = f \ \& \ \Delta(x') = x] \Longrightarrow \Delta(f'\langle x' \rangle) = f(x)$

Here $(f', x') \mapsto f'\langle x' \rangle$ is a primitive *intensional* application function on senses, just as $(f, x) \mapsto$
$f(x)$ is a primitive *extensional* application function on referents. The axiom itself leaves open

the relationship between intensional and extensional application, although we'll see later in this section, Church himself proposed that we identify them.

So we now have at least six types of entities: objects, senses of objects, concepts, senses of concepts, truth-values, senses of truth-values (which we also call *thoughts* or *propositions*). However, Frege was the first to note that there was a serious redundancy here. For, we may identify concepts with functions from objects to truth-values. (For details on this familiar identification, see circa equation (3.2) in the next section). If one does so, then one seems committed to the general claim that if a is a type of entity and b is a type of entity, then there is a type ab of entities consisting of functions from entities of type a to entities of type b. So one makes these assumptions rigorous by defining the types recursively as follows:

(2.3) (Types in the Church System) (i) there is a type e of objects, (ii) there is a type t of truth-values, (iii) if a, b are types, then there is a type ab of functions from type a entities to type b entities, and if a is a type then a' is a type of senses which present entities of type a.

In this last clause, it's important to emphasize that $a \mapsto a'$ is a primitive operation on types (cf. Kaplan [48] p. 721 and Klement [51] p. 173). Hence, a' is the result of applying an operation to the type a, and not simply another variable for types (and likewise a'' is the result of applying the prime operation to type a'). Sometimes in what follows, if we write entities of type a as $f, g, h, \ldots$ (resp. $x, y, z \ldots$), then we will adopt the convention of writing entities of type a' as $f', g', h', \ldots$ (resp. $x', y', z' \ldots$). However, under this convention, entity f' of type a' is *not* the result of applying any operation to the entity f of type a, but rather just a conventional device which allows us to visually keep track of which entity has which type.

Having set up the type system in this way, one sees immediately that there must be not a single presentation relationship Δ, but rather a presentation relationship Δ_a for each type a, which relates senses of type a' to entities of type a. Having made this distinction, one thus reformulates the Axiom that Sense Determines Reference (2.1) and the Composition Axiom (2.5) as follows:

(2.4) *Typed Sense Determines Reference*: $(\Delta_a(s, d_0) \And \Delta_a(s, d_1)) \Rightarrow d_0 = d_1$

(2.5) *Typed Composition*: $[\Delta_{ab}(f') = f \And \Delta_a(x') = x] \implies \Delta_b(f'\langle x' \rangle) = f(x)$

In the latter, the intensional application function $(f', x') \mapsto f'\langle x' \rangle$ takes an sense f' of type $(ab)'$ and a sense x' of type a' and returns a sense of type b', just as the extensional application function $(f, x) \mapsto f(x)$ takes a referent f of type ab and a referent x of type a and returns a referent of type b. Hence, just as there are as many presentation relations as there are types, so there are as many intensional and extensional application functions as there are pairs of types. Later, when we deal more formally with these systems, we will introduce symbols subscripted by types for the intensional and extensional application functions but for the time being we simply allow context to determine the types (cf. circa equation (5.14) in §5). Finally, let's record our standing assumption in this paper that entities of type ab are individuated extensionally:

(2.6) *Extensional Identity Criterion for Functional Entities*: if f, g are entities of type ab then $f = g$ if and only if $f(x) = g(x)$ for all entities x of type a.

But no analogous assumptions are made on the individuation of entities of type $(ab)'$ in this paper.

However, in his own writings, Church adopted the following axiom on the types themselves:

(2.7) *Axiom of Type Reduction*: $(ab)' = a'b'$

In other words, this axiom says that the type $(ab)'$ of senses of functions from type a entities to type b entities is identical to the type $a'b'$ of functions from senses of type a entities to senses of type b entities. It's called a reduction axiom because it allows one to reduce the senses of all higher-order entities to the senses of objects and truth-values (and senses of senses of objects, senses of propositions, etc.). The primary formal advantage of doing this is that it allows one to interpret intensional application as extensional application and hence disburdens one from developing an alternative conception of intensional application. Indeed, if senses of concepts are really extensional functions from senses of objects to senses of truth-values, then it's natural to think that the sense of a proposition is produced via the extensional application of the sense of an object to the sense of a concept.

But many of the objections to Church's intensional logic have revolved around the reduction axiom. One objection due to Dummett is based on the idea that the sense of a linguistic expression encapsulates what one knows when one knows this part of a language. Dummett observes that sometimes we learn senses of predicates by learning the sense of sentences in which they occur. But obviously such partial information won't determine a mapping from senses to senses (cf. [27] p. 293-294; [49] pp. 67 ff). Another objection due to Bealer is that functionality seems to be entirely absent from qualia and other facets of conscious experience, so that the functionality of the senses of at least some of our concept words seems unmotivated (cf. [7] p. 82, [31] p. 6).

So if, with everyone else, we reject Church's reduction axiom, then we are left with the following core of Church's system:

(2.8) The *core of Church's system* consists of the Typed Sense Determines Reference (2.4) and the Typed Composition Axiom (2.2); this theory is a typed theory, and the types are exactly as in (2.3).

So the core of Church's system is fairly sparse. In particular, there's nothing in the axioms themselves that forces or even necessarily recommends the identification of type a' with Fregean senses as opposed to any other notion of meaning. Indeed, Kaplan pointed out long ago that the standard frameworks for possible worlds semantics yield models of these axioms (cf. [48] pp. 721 ff). In particular, Kaplan proceeded by identifying type a' in the Church system with the type of functions from the worlds to the entities of type a and by defining the other primitives of the Church system as follows, wherein w_0 is fixed world (say the actual world) and w is an arbitrary world:

$$(2.9) \qquad\qquad \Delta_a(f') = f'(w_0), \qquad (f'\langle x'\rangle)(w) = (f'(w))(x'(w))$$

On the basis of these definitions, it's not too difficult to check that the Typed Sense Determines Reference (2.4) and the Typed Composition Axiom (2.2) are both satisfied. So the core of Church's system is fairly neutral on the philosophical interpretation of the intensional notions which it axiomatizes. In the next section we turn to the formalization of the Russell-Myhill paradox within this core system, and this treatment of the paradox will yield a common framework in which advocates of Fregean sense and advocates of possible worlds semantics can discuss the comparative advantages and disadvantages of different solutions to the paradox.

Before doing so, it's perhaps worth underscoring some of the departures that we have made in this paper from traditional treatments of Church's intensional logic. While there are some mere notational differences,[1] the primary non-notational differences are the following. First, Church did not introduce a primitive $(f', x') \mapsto f'\langle x'\rangle$ for intensional application; again, this was because of his adoption of the axiom on type-reduction (2.7) which we and many others reject. Further, because of his adoption of this axiom, Church did not introduce a primitive operation $a \mapsto a'$ on types but could simply get by with postulating types $o_1, o_2, \ldots, \iota_1, \iota_2, \ldots$ wherein σ_{n+1} is the type reserved for senses of entities of type σ_n for $\sigma \in \{o, \iota\}$ and where we set $o = o_1$ and $\iota = \iota_1$ (cf. [14] pp. 7, 11, [2] p. 370, [49] p. 106). If one rejects the axiom on type-reduction (2.7), then it's natural to postulate the primitive operation $a \mapsto a'$ on types, and here we follow Kaplan [48] p. 721 and Klement [51] p. 173. Finally, much of the Church's own work on his system concerned various proposals for individuating senses, and these went under the name of Alternative (0), Alternative (1), and Alternative (2) (cf. Klement [49] pp. 101 ff for overview). How exactly senses are individuated is obviously important, but it is not needed for the formalization of the Russell-Myhill Paradox discussed in the next section or for the predicative response discussed in the subsequent sections.[2]

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[1]One notational difference is that Church wrote the type reserved for functions from entities of type a to entities of type b as ba rather than ab (cf. [14] p. 12, [2] p. 370, [49] p. 106). We prefer the latter simply because it is now the norm in formal semantics (cf. [43] p. 28, [40] pp. 84, 121). Further, Church respectively used the letters o and ι instead of t and e for the truth-values and objects (cf. [14] p. 11, [2] p. 370, [49] p. 106). Again, we use t and e simply because this is now the norm (cf. [43] p. 28, [40] pp. 79, 128). Finally, sometimes in Church –and sometimes in intensional logics more generally– the word "concept," perhaps proceeded by modifiers like "individual" or "propositional," is reserved for certain kinds of senses or intensions (cf. [49] p. 96, [31] pp. 155 ff, [40] p. 122). However, here in this paper we eschew this usage and use the Fregean terminology, on which "concepts" are the unsaturated entities which may be saturated by objects and which are thus one-half of the concept-object distinction.

[2]Admittedly, there is something deeper going on here. In particular, the distinction between Alternative (0) and Alternative (1) lies in whether lambda-conversion preserves sense. However, lambda-terms are an alternative way of formalizing comprehension (cf. (4.1)) which the predicative response offered here does not have available in full generality. So the formal extensions of Church's intensional logic with which we work here simply don't have lambda-terms in the object-language. So an immediate issue which faced Church– namely whether to say that lambda-conversion preserves sense– is not even available in the object-language of our systems.

3   Formalized Version of the Russell-Myhill Paradox and Extant Responses

The aim of this section is to set out a formalized version of the Russell-Myhill Paradox and to survey some extant non-predicative responses. The formalization offered here is distinct from but owes much to the formalizations offered by Anderson and Klement, and we'll discuss explicitly in this section these similarities and differences. Since the Russell-Myhill paradox is a proposition-theoretic version of the Russell paradox about sets, it's useful to begin with a brief review of Russell's paradox about sets and Cantor's related theorem about cardinalities. Cantor's idea was to use injections and bijections to compare the cardinalities of collections. So a collection X is said to have *cardinality less than or equal* to collection Y just in case there is an injection $\iota : X \to Y$, while the two collections X, Y are said to have *the same cardinality* if there is a bijection between them.[3] Cantor's theorem about cardinalities says that for any collection, there's no injection from its subcollections to itself. So in symbols, Cantor's theorem says that for any X, there's no injection from $\{Y : Y \subseteq X\}$ to X itself. But there's a natural bijective correspondence between the subcollections $Y \subseteq X$ and the zero-one valued functions $f : X \to \{0, 1\}$, given by sending $Y \subseteq X$ to its characteristic function $f_Y : X \to \{0, 1\}$ which is defined by

$$(3.2) \qquad\qquad f_Y(x) = \begin{cases} 1 & x \in Y, \\ 0 & x \notin Y. \end{cases}$$

Since the type at is reserved for functions from entities of type a to the truth-values $\{0, 1\}$, there is thus a natural type-theoretic expression of Cantor's theorem:

(3.3) (Type-Theoretic Version of Cantor's Theorem) For any type a, there's no injection from entities of type at to entities of type a.

This version is entirely type-theoretic, since the injection in question would be an element of type $(at)a$ and since the property of being injective is expressible purely with the extensional application notions built into the type theory.

Let us briefly recall the traditional proof of the type-theoretic version of Cantor's Theorem. For the sake of readability, in this proof let us write entities of type at as $f, g, h \ldots$ and entities of type a as $x, y, z \ldots$. A function from the entities of type at to entities of type a is a function ι taking input f of type at and returning output $\iota(f) = x$ of type a. Now, suppose that there was such an injection ι from entities of type at to entities of type a. Then consider the *diagonal* map d from elements of type a to elements of type t given by

$$(3.4) \qquad\qquad d(x) = 1 \iff \exists f \ (\iota(f) = x \ \& \ f(x) = 0)$$

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[3]Recall that intuitively, injections never hit one thing with two different things, while surjections hit everything; and bijections do both of these things. More formally, one defines:

(3.1) A function $f : X \to Y$ is said to be an *injection* if $f(x_0) = f(x_1)$ implies $x_0 = x_1$ for all x_0, x_1 from X. A function $f : X \to Y$ is said to be a *surjection* if for every y from Y there is x from X such that $f(x) = y$. A function $f : X \to Y$ is said to be a *bijection* if it is both an injection and surjection.

Then let $y = \iota(d)$ and ask whether $d(y) = 1$ or $d(y) = 0$. If $d(y) = 1$, then by the left-to-right direction of equation (3.4) one obtains witness f satisfying $\iota(f) = y$ and $f(y) = 0$. Then since $\iota(d) = y = \iota(f)$, we may conclude from the injectivity of ι (cf. equation (3.1)) that $d = f$, which contradicts that $d(y) = 1$ while $f(y) = 0$. Alternatively, if $d(y) = 0$, then d and y are witness to the right-hand side of equation (3.4), and so by the right-to-left direction of this equation we have $d(y) = 1$, a contradiction. Hence, in either case we obtain a contradiction.

Sometimes Russell's paradox about sets is described as a claim to the effect that there is no set of all sets. The connection between the type-theoretic version of Cantor's theorem and Russell's paradox about sets can be made clear if one defines a membership relation by

$$(3.5) \qquad y \in x \Longleftrightarrow \exists\, f\ (\iota(f) = x\ \&\ f(y) = 1)$$

Further, for the moment let's call a *set* an entity which is in the range of the operator ι. Then for any set x, it follows from the injectivity of the ι-operator that we can also express *non*-membership in x with an existential quantifier:

$$(3.6) \qquad y \notin x \Longleftrightarrow \exists\, f\ \iota(f) = x\ \&\ f(y) = 0$$

But then it is easy to see that for sets x, equation (3.4) is equivalent to:

$$(3.7) \qquad d(x) = 1 \Longleftrightarrow x \notin x$$

So expressed in these terms, the diagonal function d from the above paragraph is the characteristic function of the collection of sets which are not members of themselves. If there was a set of all sets, then presumably this collection would form a set as well, and we then reach a contradiction by asking whether it is a member of itself or not. So this is one way to see the connection between the Russell paradox about sets and the type-theoretic version of Cantor's Theorem.

The Russell-Myhill paradox about propositions proceeds by arguing, based on considerations related directly to propositions, that there is an injection from collections of propositions to propositions. Since type t' is reserved for propositions in the Church system and since collections of propositions can be identified with their characteristic functions of type $t't$ (à la equation (3.2)), if this argument were correct than it would mean that there was an injection from entities of type $t't$ to entities of type t', which contradicts the type-theoretic version of Cantor's Theorem (3.3). So one way to respond to the Russell-Myhill paradox about propositions is to block *in a well-motivated way* the type-theoretic version of Cantor's Theorem (3.3). This is the kind of proposal which we shall pursue in this paper, beginning in §4. But in the remainder of this section we focus simply on formalizing the Russell-Myhill paradox and on surveying other extant responses.

The informal version of the Russell-Myhill paradox was initially described in Appendix B of Russell's 1903 *The Principles of Mathematics* and then again in Myhill's 1958 paper on Church's intensional logic.[4] The argument of Russell and Myhill runs as follows. Given a

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[4]More specifically see §500 p. 538 of Russell [73] and p. 82 of Myhill [59]. According to the history as set out in de Rouilhan [18], Russell never mentioned this paradox again. As for Myhill, in the same

collection of propositions $\mathcal{C}$, consider the proposition $\iota(C)$ expressed by the sentence "every proposition in $\mathcal{C}$ is true" (or "every proposition is in $\mathcal{C}$.") It seems that this function is an injection. For, suppose that $\iota(\mathcal{C}) = \iota(\mathcal{D})$. Since these two propositions differ only as to $\mathcal{C}$ and $\mathcal{D}$, then presumably the senses (or intensions) of their constituents $\mathcal{C}$ and $\mathcal{D}$ are the same as well. And this would presumably imply that $\mathcal{C}$ and $\mathcal{D}$ are the same not only in sense or intension, but that they are also the same in reference or extension. Hence the map ι is ostensibly an injection from collections of propositions to propositions. But, by applying the type-theoretic version of Cantor's Theorem (3.3), we obtain a contradiction.

While this version of the Russell-Myhill paradox is traditional, it is not obviously a formal paradox. This is for two reasons. First, formal paradoxes show that some formal system is inconsistent. But it is not at all obvious– based merely on its informal description– in what formal system the argumentation of the above paragraph may be formalized. To be sure, a good start could be made on this to the extent that one could formalize the notion of "a sentence expressing a proposition." But to the extent that one could formalize the notion of "a sentence expressing a proposition" one could presumably likewise formalize the notion of "a sentence denoting a truth-value," and hence one would worry that this formalization would require prior treatment of the liar paradox.[5] Second, formal paradoxes are always *valid* arguments, whose conclusion is that some formal set of axioms is inconsistent. But there is a real concern about the validity of the above rendition of the Russell-Myhill paradox. For, it seems at crucial points to equivocate between the collection $\mathcal{C}$ and a sense thereof (or intension thereof). Indeed, it seems that it is the latter which would contribute to the proposition expressed by "every proposition in $\mathcal{C}$ is true." Yet, the argument as a whole pertains to an injection whose domain is the collection of propositions $\mathcal{C}$. A truly formalized version of the Russell-Myhill paradox would leave no doubt as to whether the argument was, at any juncture, operating on a collection of propositions or a sense or intension thereof.[6]

The formalization of the Russell-Myhill paradox which we adopt avoids these two problems, and reads as follows:

(3.8) (*Formalized Russell-Myhill Paradox*). The following axioms are jointly inconsistent

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1958 paper he reports that Carnap's "general approach to the problem, in terms of 'possible worlds' and state-descriptions, is in [his] opinion practically certain to yield a correct explication within a few years" ([59] p. 81). This contrasts with Myhill's earlier 1952 paper on Church ([58]) in which he weighs carefully the costs and benefits of Fregean and modal approaches without indicating a decisive preference for either. It is well-known that Myhill continued to work on intuitionistic and non-classical approaches to the set-theoretic paradoxes throughout his career, but to my knowledge he never after the 1958 paper returned to this proposition-theoretic paradox.

[5]Intensional logics like Church's intensional logic and possible worlds semantics have resources for axiomatizing the notion of a "proposition denoting the true." In Church's system, this is written as $\Delta_t(p) = 1$ while in possible worlds semantics this is written $p(w_0) = 1$ where w_0 is the world of evaluation. However, in neither of these intensional logics does one have the resources for going from a name of a sentence to the proposition expressed by the sentence. If one did, then since these systems of intensional logic are consistent with the addition of resources needed to effect self-reference, one could replicate the formal versions of the liar paradox.

[6]Klement suggests that this kind of concern is one way of understanding Frege's own reservations about the Russell-Myhill paradox ([49] p. 183).

against the background of the core of Church's system (2.8): the Surjectivity Axiom (3.9), the Senses are Objects Axiom (3.10), the Propositions as Fine-Grained as Objects Axiom (3.11), and the type-theoretic version of Cantor's Theorem (3.3).

So as one can see, this formalization concerns three additional axioms which we need to introduce and motivate in these next pages prior to setting out the derivation of the paradox.

The first axiom in the formalized version of the paradox is the called the Surjectivity Axiom. In essence, this axiom says that every entity– including the higher-order ones– are presented by some sense or intension:

(3.9) *Surjectivity Axiom*: for each type a and each element f of type a, there is element f' of type a' such that $\Delta_a(f') = f$.

The immediate warrant for this axiom is that there's simply no other way to formalize the Russell-Myhill paradox. For, consider again how it opens: "for each collection of propositions $\mathcal{C}$, consider the proposition $\iota(C)$ expressed by the sentence 'every proposition in $\mathcal{C}$ is true.' " So we need some way to move from *any* collection of propositions to a proposition. It seems that any way in which we do this will take a collection of propositions, take a sense or intension which presents this collection, and build a proposition based off of this sense.[7]

The next axiom concerns the location of senses or intensions within Frege's concept-object distinction, or within the typed systems usually employed in formal semantics. In essence, it says that senses or intensions fall on the object side of the concept-object distinction:

(3.10) *Senses are Objects Axiom*: for each type a and each element f' of type a', there is element x of type e such that $f' = x$.

This axiom is non-trivial because f' and x are variables of different types. If contrary to fact they were variables of the same type, this would simply be a truth of the ambient predicate logic. There's at least two reasons for believing the Senses are Objects Axiom. First, it seems that all the arguments against the Axiom of Type Reduction (2.7) would be arguments for this axiom. For these arguments do not merely show that the senses of functions from entities of type a to entities of type b cannot be identified with functions from entities of type a' to entities of type b', but they also show that the senses of functions are themselves not functional in nature. In particular, Dummett's concern extends straightforwardly to a concern that learning a finite sample won't determine an entire function of any kind, while Bealer's concern suggests that qualia don't obviously have any functional content built into them.

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[7]The Surjectivity Axiom has a long and complicated history in Church's own writings. In 1946, Church seemed to indicate that Cantor-like paradoxes would lead one to deny this axiom ([13] p. 31). In 1974, Church indicated that this axiom followed the premises of his system called Alternative 2 ([15] p. 145). In his last paper in 1993, Church included this axiom in his system ([17] pp. 144-145), albeit without saying anything explicit about his reasons for this inclusion. For other statements of the Surjectivity Axiom in the secondary literature, see Anderson [1] principle (C) p. 221 and Klement [49] Theorem LSD(0) 1 p. 116 and Klement [50] p. 305 Axiom PC. For more on Anderson and Klement on the Surjectivity Axiom, see the discussion below.

The second kind of consideration which points in favor of the Senses are Objects Axiom is a reflection on traditional conceptions of the nature of Fregean senses: in particular, while Russell suggested that we might view Fregean senses as definite descriptions,[8] Dummett has suggested that we might understand them as certain kinds of procedures or algorithms, a "route to reference" (cf. [27] pp. 96, 102, 179 ff, [44] pp. 66 ff, [83] p. 323). If either of two traditional proposals about the nature of Fregean sense are correct, then it seems that senses ought be regarded as objects of certain kinds, as opposed to concepts: for, whatever the exact nature of definite descriptions and algorithms, presumably they are unsaturated and fall on the "object" side of Frege's concept-object distinction. For instance, if one views definite descriptions as Gödel numbers of formulas or if one views algorithms as indexes of Turing machines this will be the case.

The final axiom operative in our formalized version of the Russell-Myhill paradox (3.11) is an axiom postulating a connection between objects and propositions:

(3.11) *Propositions as Fine-Grained as Objects Axiom*: there is an injection χ from entities of type e to entities of type t'.

Since entities of type e are objects and entities of type t' are propositions, this axiom is just saying that there is an injection from objects to propositions. This axiom seems plausible in part because it follows directly from several different conceptions of propositions. If one conceives of propositions pertaining to individuals to involve definite descriptions picking out those individuals (or to procedures which hone in on these individuals), then presumably there will be such an injection: in particular, sending the individual who is *the F* to the proposition *the F is G* seems to be an injection. Alternatively, suppose that one conceives of propositions about individuals as structured entities which literally contain those individuals. On this conception, there will likewise be an injection from objects to propositions, since the map $x \mapsto$ "x is G" might be represented by a map $x \mapsto \langle x, G \rangle$ wherein $\langle \cdot, \cdot \rangle$ is the ordered pair map.

There is of course a certain commonality between the Propositions as Fine-Grained as Objects Axiom and the principle

(3.12) There's an injection from collections of propositions to propositions.

This principle of course directly violates the type-theoretic version of Cantor's Theorem (3.3). Hence, one might initially think that there is little to be gained by presenting a formalization in which the Propositions as Fine-Grained as Objects Axiom was a premise, since one might think that any reason to believe this axiom would likewise be a reason to believe the principle (3.12). But it's not obvious that this is the case. For, there's no formal relation

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[8]It is admittedly somewhat inaccurate to speak of definite descriptions merely as an "interpretation of Fregean sense," since they in fact provide a systematic way of dispensing with the Fregean notion of expression altogether and maintaining that reference is the sole semantic primitive. But presumably part of our tradition's reason for thinking that Frege's theory of meaning is susceptible to modal counterexamples couched in terms of definite descriptions is something like the thought that we can think of Fregean senses as definite descriptions.

of entailment from the Propositions as Fine-Grained as Objects Axiom itself to the principle (3.12). For instance, if the number of objects and the number of propositions are both finite and if in addition the former is much smaller than the latter, then the Propositions as Fine-Grained as Objects Axiom will be true while the principle (3.12) will be false.

Having set out and motivated the various axioms at issue in our formalized version of the Russell-Myhill paradox (3.8), let us now show that it is indeed paradoxical, that is that while the axioms are individually well-motivated, they fail to be jointly consistent. By the Surjectivity Axiom (3.9), for every collection of propositions $\mathcal{C}$, there is a sense $\mathcal{C}'$ which presents it. By the Senses are Objects Axiom (3.10), each such sense $\mathcal{C}'$ is identical to some object. So there is thus a map from collections of propositions to objects such that the object is identical to a sense which presents the collection. More formally:

(3.13) For every collection of propositions $\mathcal{C}$ there's an object x such that there is a sense $\mathcal{C}'$ such that $\mathcal{C}'$ is identical to x and $\mathcal{C}'$ presents $\mathcal{C}$.

So this induces a map $\mathcal{C} \mapsto \delta(\mathcal{C})$ from collections of propositions to objects such that

(3.14) For every collection of propositions $\mathcal{C}$ there is a sense $\mathcal{C}'$ such that $\mathcal{C}'$ is identical to $\delta(\mathcal{C})$ and $\mathcal{C}'$ presents $\mathcal{C}$.

Further, by the Propositions as Fine-Grained as Objects Axiom (3.11), there is an injection χ from objects to propositions. Let ι be the composition of the two maps, so that $\iota = \chi \circ \delta$. Then ι is a map from collections of propositions $\mathcal{C}$ to propositions $\iota(\mathcal{C})$. Further, the map ι too is an injection. For, suppose that $\iota(\mathcal{C}_1) = \iota(\mathcal{C}_2)$. Then since χ is an injection, $\delta(\mathcal{C}_1) = \delta(\mathcal{C}_2)$. Then by the characterization of δ in equation (3.14), for each $k \in \{1, 2\}$ there is sense $\mathcal{C}'_k = \delta(\mathcal{C}_k)$ which presents $\mathcal{C}_k$. But since $\delta(\mathcal{C}_1) = \delta(\mathcal{C}_2)$, we have that $\mathcal{C}'_1 = \mathcal{C}'_2$. Since $\mathcal{C}'_k$ presents $\mathcal{C}_k$, by the Typed Sense Determines Reference Axiom (2.4), it follows that $\mathcal{C}_1 = \mathcal{C}_2$. So indeed the map ι is an injection from collections of propositions to propositions, which contradicts the type-theoretic version of Cantor's Theorem (3.3).

This formalization of the Russell-Myhill paradox is distinct from but owes much to the earlier formalizations of Anderson and Klement. On the one hand, Anderson made the Surjectivity Axiom (3.9) the focus of his treatments of the paradox ([1] pp. 221 ff, [4] pp. 107 ff). However, Anderson's formalization is given in a system which includes the axiom of type-reduction (2.7), and so is not obviously available once we have rejected this axiom. In his paper [50], Klement gave a version of the Russell-Myhill paradox which invoked a "principle of conceivability" to the effect that "for every entity, there is at least one sense presenting it as referent" ([50] p. 305, cf. §5 pp. 309 ff). So our Surjectivity Axiom (3.9) is just another expression of Klement's principle of conceivability. However, in that paper, Klement worked in a system which collapsed the concept-object distinction, so that concepts were a particular species of object.[9] Since we want to work within Church's intensional logic, which is a typed system, our formalization has to proceed slightly differently. That said, one can view the formalization given above as the minimal way to modify the formalization of

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[9]See axiom "PCE" on [50] p. 305. Another way of formalizing the system of Klement [50] might be to regard it simply as an untyped system, where there is no distinction between concepts and objects.

Klement [50] into the framework of what we're calling the core of Church's system (2.8). In particular, while Klement's argument postulated that concepts were a particular species of object, the Senses are Objects Axiom (3.10) postulates that senses are a particular species of object.[10]

Let's now turn to describing extant responses to this formalized version of the Russell-Myhill paradox. Of course, the specific formalization given above is new to this paper; hence, it is not as if previous authors have explicitly addressed this specific rendition of the paradox. However, anyone who has constructed a model of the core of Church's system (2.8) has found some way to avoid this paradox, and so we can ask how these consistent formal systems evade the formalized version of the Russell-Myhill paradox. As the formalization makes clear, there are *apriori* five ways to respond to the paradox: (i) reject some aspect of the formalization Church's Core System (2.8), (ii) reject the Surjectivity Axiom (3.9), (iii) reject the Senses are Objects Axiom (3.10), (iv) reject the Propositions as Fine-Grained as Objects Axiom (3.11), (v) reject the type-theoretic version of Cantor's Theorem (3.3). While the predicativity response of the next section is a response of this last kind, in the remainder of this section we describe responses of the other four kinds.[11]

One can view the work of Pavel Tichý as an articulation of a conception of intensional entities on which the Surjectivity Axiom (3.9) comes out false. In his book *The Foundations of Frege's Logic* ([85], cf. [86], [31]), Tichý builds a theory of "constructions" that is in many ways similar to Church's notion of "presentations." It's similar enough so that Tichý's models are models of the core of Church's system. Tichý was particularly motivated by the phenomena of describing arithmetical functions with algorithms. In particular, he wanted a system in which you could both (i) evaluate $x^2 + 8$ with respect to different variable assignments and (ii) treat $x^2 + 8$ as an object. So relative to the variable assignment $x \mapsto 2$, we then say that the object "$x^2 + 8$" presents or constructs 12. Likewise, relative to the variable assignment $x \mapsto 2$, the object "$x^4 - 4$" presents or constructs 12. So Tichý built models in which the senses were just Gödel numbers of well-formed expressions but in which the higher-order variables ranged over all functions between the lower-order domains ([85] p. 66, [31] p. 44). But since there are uncountably many such functions and only countable many well-formed expressions, the surjectivity axiom will fail on Tichý's formal models. So the response to our formalized version of the Russell-Myhill paradox suggested by Tichý's work would be to say that it's a mistake to think that one can refer to every higher-order entity.

To our knowledge, Tichý is the only author who can be represented as rejecting the Surjectivity Axiom (3.9) whilst not rejecting other components of the core of Church's in-

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[10]It also bears mentioning that Church, Anderson, and Klement additionally considered formalizations of the Russell-Myhill paradox within an alternative framework of intensional logic that goes under the heading of "Russellian intensional logic" ([16], [3], [49] pp. 175 ff). Since this framework was designed to be an alternative to what we're calling "Church's intensional logic," we have not made use of this in our formalization.

[11]Of course, there's a more drastic kind of response available. This response would say that what the formal inconsistency shows is that there is no coherent notion of fine-grained meaning available and that we should hence simply purge such talk from our semantic theorizing.

tensional logic (2.8). By contrast, per Kaplan's construction described in the last section (circa equation (2.9)), we can view the standard models of possible worlds semantics as models of Church's core system, and it turns out that the Surjectivity Axiom always comes out true on these models. For, on these models, intensions are just functions from possible worlds to extensions, and so for any extension one can consider the "constant" intension that picks out that extension at any world. The models traditionally used in possible worlds semantics are so-called "standard" models in which the type ab is interpreted as the set of all functions from entities of type a to entities of type b, as judged by the ambient set-theoretic metatheory; hence, the type-theoretic version of Cantor's Theorem (3.3) also comes out true on these models. The tradition of possible worlds semantics then avoids the formalized version of the Russell-Myhill paradox by either rejecting the Propositions as Fine-Grained as Objects Axiom (3.11) or the Senses are Objects Axiom (3.10). For, if there are fewer sets of worlds than there are objects in the worlds, then of course there is no injection from propositions to objects, and so the Propositions as Fine-Grained as Objects Axiom (3.11) comes out false. Further, if there is more than one world, then by the type-theoretic version of Cantor's Theorem (3.3), there are always more functions from worlds to objects than there are objects, and so there is no way of identifying intensions of objects with objects; and hence in this situation the Senses are Objects Axiom (3.10) comes out false.

Finally, the work of Anderson (cf. [1], [2] pp. 371 ff) represents a distinct response to the Russell-Myhill paradox, on which one rethinks certain elements of Church's Core System (2.8). Anderson's basic idea was to modify Church's system so that there was not a single presentation relation Δ, but rather a series of presentation relations $\Delta^{(1)}, \Delta^{(2)}, \ldots$. Let's call $\Delta^{(n)}$ the *n-th order presentation function*. On this view, one modifies the axioms of Church's Core System (2.8) so that there is one of these axioms for each of the n-th order presentation functions. So if one wants to present a formalization of the Russell-Myhill paradox, one needs to specialize it to some specific n-th order presentation function. On this conception, it's natural to think that the analogues of the Surjectivity Axiom would be false. While it might be true in certain models that every higher-order entity was n-th order presented for some n, there would not be a single n which did this for each higher-order entity. So Anderson's response to the Russell-Myhill paradox parallels the "typed" responses to the liar paradox (cf. [2] p. 376).

This is not the place to argue against the various responses to the formalized version of the Russell-Myhill paradox which we have distilled from the writings of Tichý, Kaplan, and Anderson. For one, it seems likely that such an adjudication would ultimately proceed by reference to larger considerations like the ability of each of the resulting systems to interpret categorical grammar or to provide a satisfactory semantics for belief attributions. Further, before to trying to resolve the paradox in favor of one of these responses, it's important to understand whether we have actually exhausted the entirety of the solution space to the paradox. It seems that the predicativity response has been neglected in the extant literature on the Russell-Myhill paradox, and a chief aim of this paper, which we begin on in earnest in the next section, is to describe the general shape of a plausible predicative response to

the Russell-Myhill paradox of propositions.[12]

4 The Predicative Response to Russell-Myhill

So in the last pages we have surveyed how various constructions of models of Church's core system (2.8) avoid the formalized version of the Russell-Myhill paradox (3.8). In the extant literature there seems to have been no attempt to respond to this paradox by rejecting the type-theoretic version of Cantor's Theorem (3.3). Part of the reason for this might be that Cantor's theorem is, well, a theorem. So there might be great pressure to not reject it. But this pressure will only be so great as the strength and plausibility of the axioms from which the theorem is derived. The axioms tacit in the derivation given above of Cantor's theorem are instances of the so-called *comprehension schema*. However, there is a long tradition of *predicative mathematics*, stemming from Poincaré and Weyl and represented in our day by the likes of Feferman ([32, 33]), which proceeds by systematically restricting the comprehension schema. Our aim in what follows is simply to set out and examine a predicative response to the Russell-Myhill paradox of propositions.

Intuitively, the comprehension schema is a mechanism for converting formulas to higher-order entities. In the setting of type-theory, it's most expedient to adopt a version which says that any functional formula determines a higher-order function:

(4.1) (Typed Comprehension Schema). The *typed comprehension schema* consists of all the axioms

$$\forall\, q_1, \ldots, q_k\, [\forall\, x\, \exists!\, y\, \varphi(x, y, q_1, \ldots, q_k)] \to \exists\, h\, [\forall\, x\, \varphi(x, h(x), q_1, \ldots, q_k)]$$

where $\varphi(x, y, z_1, \ldots, z_k)$ is a formula with all free variables displayed and with free variable x of type a, y of type b, while h is a variable of type ab that does not appear free in φ.

In this schema, the exclamation notation "$\exists! x\, \theta(x)$" is simply an abbreviation for "$\exists\, x\, \theta(x)\, \&\, \forall\, z\, (\theta(z) \to z = x)$". A special case of this is the following, in which there is no requirement that the formula in question be functional in nature:

(4.2) (Concept Comprehension Schema). The *concept comprehension schema* consists of all the axioms $\forall\, q_1, \ldots, q_k\, \exists\, h\, \forall\, x\, (h(x) = 1 \leftrightarrow \psi(x, q_1, \ldots, q_k))$, where $\psi(x, z_1, \ldots, z_k)$ is a formula with all free variables displayed and with free variable x of type a, while h is a variable of type at which does not appear free in ψ.

To derive this schema from the Typed Comprehension Schema (4.1), one defines the formula $\varphi(x, y, z_1, \ldots, z_k)$ to be

(4.3) $$[(\psi(x, q_1, \ldots, q_k) \wedge y = 1) \vee (\neg\psi(x, q_1, \ldots, q_k) \wedge y = 0)]$$

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[12]Obviously other approaches have been neglected as well: for instance, I know of no extant approaches to the Russell-Myhill paradox which adopt the perspective of non-classical logic.

The reason for wanting the Typed Comprehension Schema as opposed to the mere Concept Comprehension Schema (4.2) is that one wants a way to e.g. go from the entity q of type a to the constant function $f_q(x) = q$ of type aa. Since the Concept Comprehension Schema (4.2) only delivers entities h of type at, it itself cannot do this. Further, it's worth mentioning that there's a natural principle which generalizes rather than specializes the Typed Comprehension Schema. In particular, if one removes the uniqueness clause from the antecedent of this schema, then this becomes a version of the axiom of choice:

(4.4) (Typed Choice Schema). The *typed choice schema* consists of all the axioms

$$\forall\, q_1, \ldots, q_k\ [\forall\, x\ \exists\, y\ \varphi(x, y, q_1, \ldots, q_k)] \to \exists\, h\ [\forall\, x\ \varphi(x, h(x), q_1, \ldots, q_k)]$$

where $\varphi(x, y, z_1, \ldots, z_k)$ is a formula with all free variables displayed and with free variable x of type a, y of type b, while h is a variable of type ab which does not appear free in φ.

So the Typed Choice Schema (4.4) trivially implies the Typed Comprehension Schema (4.1), which as we remarked above implies the Concept Comprehension Schema (4.2).

The predicative response to the Russell-Myhill paradox restricts the Typed Comprehension Schema (4.1) by imposing constraints on the kinds of higher-order quantifiers and higher-order parameters that can occur in the formula. These restrictions are formulated in terms of the notion of the degree of a type:

(4.5) (Degree of Type) The *degree* of a type is a positive natural number which is defined recursively as follows:

$$\|e\| = \|t\| = 1, \qquad \|a'\| = \|a\|, \qquad \|ab\| = \begin{cases} \|a\| + 1 & \text{if } \|a\| \geq \|b\|, \\ \|b\| & \text{if } \|a\| < \|b\|. \end{cases}$$

Intuitively, the idea is that degree goes up when the entities of type ab are genuinely higher-order entities. For instance, suppose that $a = t$ so that that there are only two entities of type a, namely the two truth-values 0 and 1, and suppose that $b = et$, so that there are many entities of type b, namely as many as there are concepts. Then entities of type ab are functions from $\{0, 1\}$ to concepts, and so are really just another way of talking about pairs of concepts. Hence, quantifying over entities of type ab should involve no more higher-order quantification than quantifying over concepts, and so the degree of ab should be the same as the degree of b in this case.

The predicative comprehension schema may then be defined as follows:

(4.6) (Predicative Typed Comprehension Schema). The *predicative typed comprehension schema* consists of all the axioms

$$\forall\, q_1, \ldots, q_k\ [\forall\, x\ \exists!\, y\ \varphi(x, y, q_1, \ldots, q_k)] \to \exists\, h\ [\forall\, x\ \varphi(x, h(x), q_1, \ldots, q_k)]$$

where $\varphi(x, y, z_1, \ldots, z_k)$ is a formula with all free variables displayed and with free variable x of type a, y of type b, while h is a variable of type ab which does not appear free in φ, and in addition variable z_i has type c_i with $\|c_i\| \leq \|ab\|$ and all the bound variables in $\varphi(x, y, z_1, \ldots, z_k)$ have type c with $\|c\| < \|ab\|$.

Using the same trick as above in equation (4.3), it's easy to see that the Predicative Typed Comprehension Schema (4.6) implies a version of the Concept Comprehension Schema in which there are the same restrictions on the parameters and bound variables appearing in the formula. For the sake of completeness, we state this version here:

(4.7) (Predicative Concept Comprehension Schema). The *predicative concept comprehension schema* consists of all the axioms

$$\forall\, q_1, \ldots, q_k\ \exists\, h\ \forall\, x\ (h(x) = 1 \leftrightarrow \psi(x, q_1, \ldots, q_k))$$

where $\psi(x, z_1, \ldots, z_k)$ is a formula with all free variables displayed and with free variable x of type a, while h is a variable of type at which does not appear free in ψ, and in addition variable z_i has type c_i with $\|c_i\| \leq \|a\| + 1$ and all the bound variables in $\psi(x, z_1, \ldots, z_k)$ have type c with $\|c\| < \|a\| + 1$.

Intuitively, this schema says that every formula in a free variable determines a higher-order entity, provided that the higher-order quantifiers in the formula not have a type with degree which exceeds that of the type of the free variable of the formula itself. Finally, one has the predicative version of the Typed Choice Schema (4.4):

(4.8) (Predicative Typed Choice Schema). The *predicative typed choice schema* consists of all the axioms

$$\forall\, q_1, \ldots, q_k\ [\forall\, x\ \exists\, y\ \varphi(x, y, q_1, \ldots, q_k)] \to \exists\, h\ [\forall\, x\ \varphi(x, h(x), q_1, \ldots, q_k)]$$

where $\varphi(x, y, z_1, \ldots, z_k)$ is a formula with all free variables displayed and with free variable x of type a, y of type b, while h is a variable of type ab which does not appear free in φ, and in addition variable z_i has type c_i with $\|c_i\| \leq \|ab\|$ and all the bound variables in $\varphi(x, y, z_1, \ldots, z_k)$ have type c with $\|c\| < \|ab\|$.

Again, it's easy to see that the Predicative Typed Choice Schema (4.8) is the deductively strongest schema, so that it implies the Predicative Typed Comprehension Schema (4.6), which in turn implies the Predicative Concept Comprehension Schema (4.7).

Let's illustrate these predicative schemata by reference to the derivation of the type-theoretic version of the Cantor's Theorem (3.3). In particular, recall the following equation where we defined the diagonal function d of type at:

(3.4) $$d(x) = 1 \iff \exists\, f\ (\iota(f) = x\ \&\ f(x) = 0)$$

This is a non-predicative instance of the concept comprehension schema. For, while the defining formula of d has free variable x of type a with degree $\|a\|$, this formula at the same time contains a bound variable f of type at. But by consulting the definition of the degree of a type (4.5), we see that $\|at\| > \|a\|$, so that the defining equation (3.4) of the diagonal function d is not an instance of the Predicative Concept Comprehension Schema (4.7), even though it is an instance of the more general Concept Comprehension Schema (4.2). Hence,

this illustrates how if one accepts only the Predicative Typed Comprehension Schema (4.6), the traditional proof of the type-theoretic version of the Cantor's Theorem (3.3) is blocked.[13]

So the predicativity response to the Russell-Myhill paradox has two great burdens. First, it is necessary to show that the Predicative Typed Comprehension Schema (4.6) is consistent with the remaining axioms from the formalized version of the Russell-Myhill paradox (3.8). In the next section, we discharge this burden by proving:

(4.12) (Predicative Consistency Theorem). The following formal system is consistent: the core of Church's system (2.8), the Surjectivity Axiom (3.9), the Senses are Objects Axiom (3.10), the Propositions as Fine-Grained as Objects Axiom (3.11), and the Predicative Typed Choice Schema (4.8).

The second burden of the predicativity response is to say something about what motivates the restriction on the Typed Comprehension Schema (4.1). Again, intuitively this schema says that every functional formula determines a higher-order entity. If one restricts this, one must say something about *when* and *why* a functional formula determines a higher-order entity. In the remainder of this section, we discuss this more philosophical dimension of the predicativity response to the Russell-Myhill paradox.

Poincaré and Weyl were the original predicativists. They drew attention to the fact that higher-order definitions are not in general preserved when one keeps the first-order domain

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[13]A similar elementary observation can be used to illustrate the motivation for the constraints on the parameters in the Predicative Comprehension Schema (4.7). In particular, if we did not have these restrictions, then we could again show that diagonal function d from (3.4) was in fact a higher-order entity. In addition to our earlier conventions, in this paragraph let us use α, β, γ to range over entities of type aa. Then consider the admittedly uninteresting-appearing formula $\theta(x, y, z) \equiv y = z$. Let q be of type a. Then trivially we have $\forall x \exists ! y \, \theta(x, y, q)$. Then by the Predicative Comprehension Schema (4.6), there is a function α_q of type aa such that $\alpha_q(x) = q$ for all x of type a. Let $\Theta(q, \alpha) \equiv \forall x \, \alpha(x) = q$. Then by the arguments given so far in this paragraph, we have that $\forall q \exists ! \alpha \, \Theta(q, \alpha)$. Hence again by the Predicative Comprehension Schema (4.6), there is a function $\mathcal{C}$ of type $a(aa)$ such that for all q of type a, we have $\mathcal{C}(q) = \alpha_q$. Now, again by the Predicative Concept Comprehension Schema (4.7), consider the following "higher-order diagonal" $\mathcal{D}$ function of type $(aa)t$:

(4.9) $$\mathcal{D}(\alpha) = 1 \Longleftrightarrow [\exists q \, \alpha = \mathcal{C}(q) \, \& \, \exists f \, \iota(f) = \alpha(q) \, \& \, f(\alpha(q)) = 0]$$

Intuitively, $\mathcal{D}$ is picking out those constant functions α_q such that $d(q) = 0$ (where d is the diagonal function from equation (3.4)). Now, $\mathcal{D}$ has type $(aa)t$ with degree $\|(aa)t\| > \|a\| + 1$. So if, contrary to fact, there were no restrictions on parameters in the Predicative Concept Comprehension Schema (4.7), then we could use $\mathcal{D}, \mathcal{C}$ to define a higher-order entity $\widetilde{d}$ of type at as follows:

(4.10) $$\widetilde{d}(\widetilde{q}) = 1 \Longleftrightarrow \mathcal{D}(\mathcal{C}(\widetilde{q}))$$

Then one can verify that

(4.11) $$\widetilde{d}(\widetilde{q}) = 1 \Longleftrightarrow \exists f \, (\iota(f) = \widetilde{q} \, \& \, f(\widetilde{q}) = 0)$$

from which it follows that we have shown that the diagonal function d (3.4) again exists as a higher-order entity. This is why it is necessary to include restrictions on parameters in the Predicative Typed Comprehension Schema (4.6).

fixed but expands the range of the higher-order quantifiers. For instance, in describing Poincaré's conception of predicativity, Chihara tells us that:

> [...] predicative classifications are those that cannot be "disordered" by the introduction of new elements; non-predicative classifications are those that are altered or disordered by the introduction of new elements ([11], p. 47).

Likewise, in his 1918 *Das Kontinuum*, Weyl draws attention to the fact that if one codes real numbers and continuous functions as certain sets of natural numbers (or rationals), then the definition of the class of continuous functions will contain higher-order quantifiers and thus what counts as a continuous function will depend crucially on the extent and range of the higher-order quantifiers:

> If we regard the principles of definition as an "*open*" system, i.e., if we reserve the right to extend them when necessary by making additions, then in general the question of whether a given function is continuous must also remain *open* [...]' ([92] p. 66).

The dual to Weyl's remark is that if one wants to define notions whose extension does not vary with the extent and range of the higher-order quantifiers, then one should restrict attention to definitions which do not contain higher-order quantifiers.

In my view the best motivation for the predicativity restriction is related to these original thoughts of Poincaré and Weyl.[14] In more modern terms, these ideas might be expressed in terms of intuitions about the stability of reference. Suppose that one is using a definite description to refer to an object. If minor variations in empirical conditions cause the object to fail to satisfy this description, then the definite description will be a less than efficacious route to reference. There's a natural generalization of this line of thought to the comprehension schema, where for the sake of simplicity we focus on the Concept Comprehension Schema (4.2). The idea is that there's a natural way of seeing each instance of this schema as related to a definite description of a higher-order entity. In particular, consider the following instance of the Concept Comprehension Schema (4.2):

$$(4.13) \qquad \exists\, h \,\forall\, x \,(\varphi(x) \leftrightarrow h(x) = 1)$$

One can think about this h as "the $\widetilde{\varphi}$," where we define:

$$(4.14) \qquad \widetilde{\varphi}(h) \equiv [\forall\, x \,(\varphi(x) \leftrightarrow h(x) = 1)]$$

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[14]A distinct set of motivations for predicativity constraints come from the apparent affinity of predicativity with types of constructivism. For more on this complicated aspect of the history of predicativity, see Parsons [62]. Another important study of the history of predicativity-like conceptions is Goldfarb's [41] study of Russell. Goldfarb suggests that Russell's reasons for endorsing predicativity-like constraints might be related to having systems in which one can quantify over intensional entities like propositions. Suffice it to say that the motivations given here for predicativity constraints are not intended to have anything to do with constructivity and are intended to apply with equal force to the quantifiers ranging over intensional entities like propositions as to those ranging over extensional entities like concepts.

If the formula $\varphi(x)$ contains higher-order quantifiers, then whether a given h satisfies the description $\widetilde{\varphi}(h)$ may vary with expansions of the range of the higher-order quantifiers. However, when the formula does not itself contain higher-order quantifiers, then whether something satisfies this description will be stable under expansions of the range of the higher-order quantifiers.

So the motivation for the Predicative Typed Comprehension Schema (4.6) can then be seen to derive from the intuition that where one uses a definite description to effect reference to a higher-order entity, this description should be stable under variations of the range of the higher-order quantifiers. There is nonetheless still a residual philosophical challenge for this predicative response. In particular, it must say something about what the pre-theoretic idea is behind the relevant sense of expansion of the range of the higher-order quantifiers. In my view, the best answer to this is tied to the kinds of positive reasons we can give for the Surjectivity Axiom (3.9). The best positive reason to believe this axiom flows from a conception of what we're trying to model: we're not trying to model higher-order entities as they are in some abstract inaccessible third realm, but we're trying to model higher-order entities insofar as they are within our referential ken. Given this, it's natural to think that our resources for referring to higher-order entities expands over time just as our resources for referring to concrete objects expands over time.[15]

However, it seems safe to say that the same motivations in terms of stability of reference cannot be given for the more general Predicative Typed Choice Schema (4.8). Like the Predicative Typed Comprehension Schema (4.6), instances of this schema are conditionals which articulate a sufficient condition for the existence of a higher-order entity. However, unlike the Predicative Typed Comprehension Schema (4.6), it does not seem that the sufficient condition offered by the Predicative Typed Choice Schema (4.8) can be conceived of as providing any sort of intension which may serve as a mechanism by which to effect reference to the higher-order entity in question. So the Predicative Typed Choice Schema (4.8) should not be viewed as following from the predicative viewpoint articulated here, but rather should be viewed as a consistent tool which one can consistently avail oneself of.

This is important to be clear about because it's only in the presence of some choice-like principle that the Formalized Version of the Russell-Myhill Paradox (3.8) is actually a deductively valid argument. In particular, in the derivation of the inconsistency in §3, it's easy to see that the move from equation (3.13) to equation (3.14) is an instance of some choice-like principle, and one can verify that this move will be covered by the Predicative Typed Choice Schema (4.8) (and hence the Typed Choice Schema (4.4)). So if one wanted

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[15]It's natural to then ask whether this motivation for the restriction on the quantifiers in the Predicative Typed Comprehension Schema (4.6) likewise motivates the restriction on the higher-order parameters in this schema. In this it's helpful to recall the worked-out example above of the higher-order parameters $\mathcal{D}$ and $\mathcal{C}$. As one can see by inspecting equations (4.10) and (4.11) above, higher-order parameters are able to go proxy for higher-order quantifiers. Given this, if one wants to employ a description featuring a higher-order parameter to stably refer to a lower-order entity, then it's natural to require that this higher-order parameter likewise not shift in extension under variations of the range of the higher-order quantifiers. This of course suggests a certain conception of higher-order parameters: we should think about them less as static entities with a fixed extension and more as reifications of certain definitions, so that they will change whenever the range of the quantifiers in terms of which they are defined changes.

to be very formal, another axiom that should be officially added to the list of the Formalized Version of the Russell-Myhill Paradox (3.8) should be the Typed Choice Schema (4.4). So on this way of putting the matter, the predicative response to the paradox is to deny the type-theoretic version of the Cantor's Theorem (3.3) and to remain ambivalent on the Predicative Typed Choice Schema (4.8). Again, by the results of the next section, it's consistent for the predicative response to assume the Predicative Typed Choice Schema (4.8), but what we have sought to emphasize in these paragraphs is that the reasons which motivate the predicative restrictions on the comprehension schema don't obviously motivate any instances of either the impredicative or predicative choice schema.

5   The Consistency of the Predicative Response

In this section, we take up the task of demonstrating the Predicative Consistency Theorem (4.12). The reader who is uninterested in this result or merely willing to accept it conditionally might elect to pass on directly to the next section, since the precise materials developed in this section will not be needed for understanding the subsequent sections of this paper. The most fundamental idea in the proof is to replace the use of the cumulative hierarchy of sets as deployed in the usual model theory of type systems with the constructible hierarchy of sets. Before recalling the definitions of the cumulative hierarchy and the constructible hierarchy of sets, let's then first recall how the usual model theory for type theory proceeds. So let's restrict attention to the *extensional* fragment of the types which contains the type e for objects, the type t for truth-values, and which contains the type ab of functions from entities of type a to entities of type b whenever it contains type a and type b; and moreover, let's consider the *extensional* language which is bereft of the presentation symbols and the intensional application symbols and which contains only the extensional application symbols.

Models for the extensional fragment of type theory usually begin with an assignment of domains D_a to each type a. The procedure here is that the type e of objects is assigned some arbitrary domain $D_e = E$, the type t of truth-values is assigned the set $D_t = \{0, 1\}$ of truth-values (0 for "false" and 1 for "true"), and the type ab is assigned the domain D_{ab} of all functions $f : D_a \to D_b$, which is sometimes written in exponential notation as $D_b^{D_a}$ (cf. [43] p. 28, [40] pp. 84, 121):

$$(5.1) \qquad D_e = E, \qquad D_t = \{0, 1\}, \qquad D_{ab} = D_b^{D_a} = \{f : D_a \to D_b\}$$

To see the connection with the cumulative hierarchy of sets, recall that we can identify sets with their characteristic functions (cf. equation (3.2)). Further, recall that

(5.2) *The power set $P(X)$ of a given set X is defined to be the set of all the subsets Y of X, that is, $P(X) = \{Y : Y \subseteq X\}$.*

Hence, we can identify the sets in $P(D_a)$ with the functions in D_{at}. For the moment, let's write $D_{at} \approx P(D_a)$ as a shorthand for this identification. Iterating this, we can build the

following sequence in a very straightforward manner:

$$(5.3) \qquad D_e = E, \quad D_{et} \approx P(E), \quad D_{(et)t} \approx P(P(E)), \quad D_{((et)t)t} \approx P(P(P(E))), \ldots$$

So from this perspective, the usual model theory for the extensional theory of types is closely related to iterations of the powerset operator.

This kind of sequence is of course also built into the standard conception of the set-theoretic universe, namely the cumulative hierarchy. In particular, the axioms of set theory guarantee that the universe V of sets is identical to the union of the following sequence of sets V_α, where α is an ordinal (cf. [53] p. 95, [46] p. 64, [45] p. 257):

$$(5.4) \qquad V_0 = \emptyset, \qquad V_{\alpha+1} = P(V_\alpha), \qquad V_\alpha = \bigcup_{\beta < \alpha} V_\beta, \ \alpha \text{ limit}$$

One of Gödel's many important innovations in set theory was the definition of the constructible hierarchy of sets. This hierarchy is identical to the cumulative hierarchy except at the successor steps, where instead of looking at the full powerset of the previous step, one looks at a certain class of definable subsets of the previous step. In particular, we define, in contrast to the definition of the powerset (5.2) above:

(5.5) *The collection of definable subsets* $\mathrm{Defn}(X)$ *of a given set is defined to be the set of all subsets* Y *of* X *such that there is a first-order formula in the language of set theory* $\varphi(x, z_1, \ldots, z_n)$ *with all free variables displayed and parameters* $q_1, \ldots, q_n$ *from* X *such that* $Y = \{x \in X : (X, \in) \models \varphi(x, q_1, \ldots, q_n)\}$.

So for instance X is in $\mathrm{Defn}(X)$ since $X = \{x \in X : (X, \in) \models x = x\}$ and the empty set $\emptyset$ is in $\mathrm{Defn}(X)$ since $\emptyset = \{x \in X : (X, \in) \models x \neq x\}$. So Gödel then defined the *constructible hierarchy* as follows (cf. [53] p. 166, [54] p. 134, [46] p. 174, [23] p. 58):

$$(5.6) \qquad L_0 = \emptyset, \qquad L_{\alpha+1} = \mathrm{Defn}(L_\alpha), \qquad L_\alpha = \bigcup_{\beta < \alpha} L_\beta, \alpha \text{ limit}$$

and he defined the *constructible universe* L to be the union of the sets from the constructible hierarchy. Gödel further showed that the constructible universe models all of the axioms of set theory, so that this is not just another collection of sets but an alternative set theoretic universe.

The key idea in our proof of the Predicative Consistency Theorem (4.12) is to assign the types to levels of the constructible hierarchy. To do this, we need to work with a very specific kind of level of the constructible hierarchy. This kind of level was first defined by Kripke ([52]) and Platek ([65]), who had the idea that some initial segments of the constructible hierarchy can't "tell" that they are tall, and actually think that they can be "shrunk", and this idea was later famously employed by Jensen ([47]) in his proof of the uniformization theorem. So formally one defines this most fundamental notion as follows (cf. [47] pp. 256-257, [77] Definition 2.1 p. 619, [23] p. 156, [76] p. 157, Barwise [6] Definition V.6.1 p. 174, [52] p. 162, [88]):

(5.7) the *n-th projectum* $\rho_n(\alpha)$ is the smallest $\rho \leq \alpha$ such that there is a $\Sigma_n^{L_\alpha}$-definable injection $\iota : L_\alpha \to \rho$.

In this, Σ_n-definability is first-order definability in the sense at issue in the definition of $\mathrm{Defn}(X)$ above in equation (5.5), but restricted to first-order formulas which begin with a block of alternating quantifiers of length n starting with an existential quantifier (and allowing parameters). Further, when not clear from context, one writes $\Sigma_n^{L_\alpha}$ to emphasize that the definability is with respect to the structure L_α.

So now we can proceed to a description of our models. Our most important definition is the following definition of an *intensional position* (5.8). The motivation for this name comes in the subsequent definition of an *intensional hierarchy* (5.9) which is given by a collection of *intensional positions*, which intuitively are "positions" for the higher-order variables within the hierarchy.[16] While the definition of an intensional position is admittedly complicated, the broader significance of each element of this definition will be borne out by the subsequent discussion in this section:

(5.8) An *intensional position* $\mathfrak{p}$ is a given by a sextuple $\mathfrak{p} = (\alpha, \ell, \iota, \mathcal{O}, \pi, \nu)$ wherein (i) α is a limit ordinal, (ii) ℓ is a positive natural number such that L_α is a model of the Σ_ℓ-collection schema and the $\Sigma_{\ell-1}$ separation schema, (iii) the ordinal α has nontrivial ℓ-th projectum $\alpha_0 = \rho_\ell(\alpha) < \alpha$ with $\iota : L_\alpha \to \alpha_0$ a witnessing $\Sigma_\ell^{L_\alpha}$-definable injection, (iv) the set $\mathcal{O}$ is a $\Sigma_\ell^{L_\alpha}$-definable subset of α_0, (v) the map $\pi : \mathcal{O} \dashrightarrow L_\alpha$ is a $\Sigma_\ell^{L_\alpha}$-definable partial surjective function such that $\pi \circ \iota$ is the identity on L_α and such that $\mathcal{O} \setminus \pi^{-1}(L_\alpha)$ is $\Sigma_\ell^{L_\alpha}$-definable and (vi) the definability in each of the previous items is with respect to the parameter ν from L_α.

In part (ii) of this definition, the Σ_n-collection schema is the axiom schema which says that if $\varphi(x, y)$ is a Σ_n-formula and for all $x \in a$ there is y such that $\varphi(x, y)$, then there is a set b such that for all $x \in a$ there is $y \in b$ with $\varphi(x, y)$. In essence, Σ_n-collection says that when for everything in an antecedently specified set a there is a witness to a Σ_n-condition, then at least one witness for everything in a may be bounded or collected together in another set b. The Σ_n-*separation schema* is simply the separation schema from the ambient set theory restricted to the case of Σ_n-formulas: it says that if $\varphi(x)$ is a Σ_n formula and a is a set then there is another set b such that $z \in b$ iff $z \in a \land \varphi(z)$. So in essence, Σ_n-separation just says that all the Σ_n-subsets of antecedently specified a set exist. In the case $n = 1$, the structure L_α satisfies Σ_n-collection and Σ_{n-1}-separation just in case it is a model of Kripke-Platek set theory ([52], [65], Devlin [23] p. 48, p. 36). (For all $n \geq 1$, this theory was called KP_n in [88]).

Having all this in place, we may now define the notion of an *intensional hierarchy*:

(5.9) An *intensional hierarchy* $D = (\mathfrak{p}_1, \mathfrak{p}_2, \ldots)$ is given by a countable sequence $\mathfrak{p}_n = (\alpha_n, \ell_n, \iota_n, \mathcal{O}_n, \pi_n, \nu_n)$ of intensional positions such that (i) for all $n, m \geq 1$ it is the

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[16]The language of "positions" is apt because, as one can see from inspection of the below definitions, one intensional position can provide the interpretation of the n-th order quantifiers in one intensional hierarchy but the interpretation of the m-th order quantifiers in another.

case that $\rho_{\ell_n}(\alpha_n) = \rho_{\ell_m}(\alpha_m) = \alpha_0$, and (ii) the associated sequence of ordinals is strictly increasing: $\alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < \alpha_{n+1} < \cdots$.

There are two kinds of examples of intensional positions and intensional hierarchies, which are closely related to the two traditional interpretations of Fregean sense discussed above in §3 (shortly after equation (3.10)). The first example is related to algorithms. Let $X \subseteq \omega$ be an arbitrary subset of natural numbers; then one can consider Turing computability using X as an oracle (cf. [82] Chapter 3). Let $\alpha = \omega_1^X$ be the least ordinal not computable in oracle X. Then the partial function $\pi : \omega \dashrightarrow L_\alpha$ which sends Gödel numbers for formulas and indexes for X-computable ordinals to the relevant member of L_α is partial surjective, from which one can obtain a (right)-inverse injection $\iota : L_\alpha \to \omega$, from which it follows that $\rho_1(\alpha) = \omega$. This construction results in an intensional position. One can then obtain an intensional hierarchy by choosing successively more complex subsets of natural numbers $X_1, X_2, \ldots$ so that the sequence $\omega_1^{X_1} < \omega_1^{X_2} < \cdots$ is strictly increasing. (For more details, see [88], and in particular the theorem on the classical results on the first non-computable ordinal).

The other example is related to definite descriptions. Let λ be a cardinal in Gödel's constructible universe L, and let $\kappa = \lambda^+$ be the next biggest cardinal in L, as judged by L. Further, let $M_n = \mathrm{dcl}_\Sigma^{L_\kappa}(\lambda \cup \{\lambda\})$ be the sets in L_κ that have Σ_n-definite descriptions over L_κ with parameters from $\lambda \cup \{\lambda\}$. Then it can be shown that that $M_n = L_{\alpha_n}$ for some α with $\lambda < \alpha_n < \alpha_{n+1} < \kappa$ and that $\rho_n(\alpha_n) = \lambda$. For the latter, the partial function $\pi : \lambda \dashrightarrow L_\alpha$ which sends a Gödel number for a formula and its parameters to a set in L_α satisfying the formula is surjective, and so one can obtain a (right)-inverse injection $\iota : L_\alpha \to \lambda$. As Myhill put the intuitive point in his study of the related notion of the hereditarily ordinal definable sets, "One often hears it said that since there are indenumerably many sets and only denumerably many names, therefore there must be nameless sets. The above shows this argument to be fallacious."[17] So what these two examples from this and the previous paragraph tell us is that there are two canonical ways to shrink an initial segment of the constructible hierarchy: in the case where everything is algorithmic, we can shrink by sending a set to an algorithm which codes it, while in the case where everything has definite descriptions, we can send a set to a code for one of its definite descriptions.

Each intensional hierarchy D naturally gives rise to a model of Church's core system (2.8). In particular, we assign types to domains as follows:

$$(5.10) \qquad D_e = \alpha_0, \quad D_t = \{0, 1\}, \quad D_{ab} = D_b{}^{D_a} \cap L_{\alpha_{\|ab\|}}, \quad D_{a'} = \mathcal{O}_{\|a\|}$$

In this, recall that $a \mapsto \|a\|$ is the degree of the type a, as defined in (4.5). So the parallel to the usual semantics for extensional type theory becomes vivid. In particular, whereas these usual semantics employ the cumulative hierarchy to assign domains to types, here our semantics for our intensional type theory uses the constructible hierarchy to assign domains

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[17]See Myhill [57] p. 981. The hereditary ordinal definable sets, or HOD, is defined to be the class of sets x such that x has a definite description relative to the ordinals, as does every one of its members of x, and the members of members of x, etc. For a formal definition, see [53] p. 161, [54] p. 146, [46] p. 196. For more details on the construction described in this paragraph, see [88], and in particular the existence theorem.

to types. For instance, instead of assigning ab the set $D_b^{D_a} = \{f : D_a \to D_b\}$, we only assign it those elements of this set which are in the constructible hierarchy at an appropriate level. So the intuitive thought is that we're only putting those higher-order entities of type ab in the range of the higher-order quantifiers when it has entered a level of the constructible hierarchy which is coordinated with the degree of the type ab.

Having defined intensional hierarchies, our next goal is to say how to interpret the extensional application symbols $(f, x) \mapsto f(x)$, the presentation symbols Δ_a, and the intensional application symbols $(f', a') \mapsto f'\langle x'\rangle$ on intensional hierarchies. In providing these interpretations, we shall be associating each intensional hierarchy D to an *intensional structure* $\mathbb{D}$ augmented by these interpretations. Further, as we go along, we shall also show that various axioms are true on these intensional structures. However, prior to doing this, we need to state the following elementary result about how the domains D_a of an intensional hierarchy relate to the sets L_{α_n}:

(5.11) For all $n \geq 1$, both of the following hold:

> (I) for all types a with $\|a\| < n$, there is a Σ_1-formula in parameter μ_n such that D_a is the unique element of L_{α_n} which satisfies this formula, wherein μ_n is defined by $\mu_n = \langle \nu_1, \ldots, \nu_n, \alpha_0, \alpha_1, \ldots, \alpha_{n-1}\rangle$.
>
> (II) for all types a with $\|a\| = n$, the set D_a is a $\underset{\sim}{\Sigma}_{\ell_n}$-definable subset of L_{α_n} in parameter μ_n.

For a proof, see Appendix 1 §9. This result is important because it tells us that the domain D_a is a subset of $L_{\alpha_{\|a\|}}$, so that we can locate the domain D_a amongst the levels of the constructible hierarchy by calculating the degree of the type a. Further, from this we can deduce the following:

(5.12) (Proposition on Domain and Codomain of Projectum Witnesses) For all types a, one has that restriction $\iota_{\|a\|} \restriction D_a$ has domain D_a and codomain $D_{a'}$, i.e. $\iota_{\|a\|} \restriction D_a : D_a \to D_{a'}$.

To see this, let $n = \|a\|$. By the Proposition on the Location of Domains, D_a is a subset of the domain L_{α_n} of the injection $\iota_n : L_{\alpha_n} \to \alpha_0$. So the restriction notation $\iota_{\|a\|} \restriction D_a$ makes good sense. Suppose now that x is a member of D_a and set $y = \iota_{\|a\|}(x)$. Since $\pi_n \circ \iota_n$ is the identity function on L_{α_n}, we have that y is in the domain of $\pi_{\|a\|}$, which by definition is a subset of $\mathcal{O}_{\|a\|} = D_{a'}$. This, in any case, is the elementary argument which characterizes the Domain and Codomain of the Projectum Witnesses (5.12).

Given an intensional hierarchy D, there's a natural interpretation of the presentation symbols Δ_a such that the Typed Sense Determines Reference Axiom (2.4) is true on the induced intensional structure $\mathbb{D}$. In particular, the presentation functional Δ_a is interpreted on an intensional hierarchy D as the binary relation on $D_{a'} \times D_a$ defined by

$$(5.13) \qquad\qquad \Delta_a(f', f) \iff \pi_{\|a\|}(f') = f$$

That is, Δ_a is interpreted as the graph of $\pi_{\|a\|}$ restricted to $D_{a'} \times D_a$. This definition makes good sense. For, suppose that f' is from $D_{a'} = \mathcal{O}_{\|a\|}$ and f is from D_a. Then

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by the Proposition on Location of Domains (5.11), we have that $f \in D_a \subseteq L_{\alpha_{\|a\|}}$ and by definition $\pi_{\|a\|} : \mathcal{O}_{\|a\|} \dashrightarrow L_{\alpha_{\|a\|}}$.

So we just showed how to expand an intensional hierarchy D to an intensional structure $\mathbb{D}$ which has an interpretation of the presentation symbols. Now let us verify that the Typed Sense Determines Reference Axiom (2.4) is true on the intensional structure $\mathbb{D}$. For the ease of readability, we reproduce this axiom here:

(2.4) *Typed Sense Determines Reference*: $(\Delta_a(f', f) \,\&\, \Delta_a(f', g)) \Longrightarrow f = g$

So suppose that $\Delta_a(f', f)$ and $\Delta_a(f', g)$. Then by definition in equation (5.13), we have that $\pi_{\|a\|}(f') = f$ and $\pi_{\|a\|}(f') = g$. Since $\pi_{\|a\|} : \mathcal{O}_{\|a\|} \dashrightarrow L_{\alpha_{\|a\|}}$ is a partial function, it then follows that $f = g$. Since the Typed Sense Determines Reference Axiom (2.4) comes out true on this interpretation of the presentation symbols, we have that the presentation symbol is functional. Hence, just as when working in the object language of Church's core system (2.8), instead of writing $\Delta_a(f', f)$, we shall write $\Delta_a(f') = f$. Likewise, we shall write $\Delta_a(f')\!\downarrow$ to indicate that there is f such that $\Delta_a(f') = f$ (cf. discussion subsequent to (2.1) in §2).

So it remains to indicate the interpretation of the extensional application symbols $(f, x) \mapsto f(x)$ and the intensional application symbols $(f', a') \mapsto f'\langle x' \rangle$. The extensional application symbols are comparatively straightforward: these are interpreted as the function from $D_{ab} \times D_a$ to D_b given by the notion of extensional application from the metatheory. This makes sense because, per the definition of D_{ab} in equation (5.10), every element of D_{ab} is a function $f : D_a \to D_b$. It's perhaps worth underscoring that for each pair of types a, b, there is a separate extensional application symbol in the signature of Church's core system (2.8). We can usually ignore this since their interpretation is uniform. But, in what follows, if we need to explicitly display the types of an extensional application symbol, we shall write e-app$_{ab}(f, x)$ instead of $f(x)$ for the extensional application symbols.

Likewise, we shall sometimes write i-app$_{ab}(f', x')$ instead of $f'\langle x' \rangle$ for the intensional application symbols, again to highlight the fact that there is one of these symbols for each pair of types a, b. We interpret these symbols on an intensional hierarchy as follows:

(5.14) $$\text{i-app}_{ab}(f', x') = f'\langle x' \rangle = \iota_{\|b\|}((\Delta_{ab}f')(\Delta_a x'))$$

From what we know about the interpretation of the presentation functions and the result on the Domain and Codomain of the Projectum Witnesses (5.12), we see that the intensional application function is a partial function i-app$_{ab} : D_{(ab)'} \times D_{a'} \dashrightarrow D_{b'}$. So as with the discussion of the presentation functions, technically in the formal system we shall identify the intensional application function with its graph, which is a ternary relation on $D_{(ab)'} \times D_{a'} \times D_{b'}$. As with presentation symbols, when we write i-app$_{ab}(f', x')$ all by itself, it is assumed that this is defined.

Now, let's show that the Typed Composition Axiom (2.5) comes out true on this interpretation. For the ease of readability, we reproduce this axiom here:

(2.5) *Typed Composition*: $[\Delta_{ab}(f') = f \,\&\, \Delta_a(x') = x] \Longrightarrow \Delta_b(f'\langle x' \rangle) = f(x)$

So suppose that $\Delta_{ab}(f') = f$ and $\Delta_a(x') = x$. Then by its definition in equation (5.14), we see that $f'\langle x' \rangle$ is defined. Then we may evaluate the term $\Delta_b(f'\langle x' \rangle)$ as follows:

$$(5.15) \qquad \Delta_b(\iota_{\|b\|}((\Delta_{ab}f')(\Delta_a x'))) = \Delta_b(\iota_{\|b\|}((f)(x))) = (\pi_{\|b\|} \circ \iota_{\|b\|})(f(x)) = f(x)$$

where the last equality follows from the fact that $\pi_{\|b\|} \circ \iota_{\|b\|}$ is the identity function on the set $L_{\alpha_{\|b\|}}$ (cf. clause (v) in the definition of an intensional position (5.8)). So this is why the Typed Composition Axiom (2.5) comes out true on intensional structures.

Finally, let's note why the Surjectivity Axiom (3.9) and the Senses are Objects Axiom (3.10) are rather trivially true on intensional structures. As for surjectivity, suppose that f is an element of domain D_a. Again, by the Proposition on Location of Domains (5.11), we have that D_a is a subset of $L_{\alpha_{\|a\|}}$. Since $\pi_{\|a\|} : \mathcal{O}_{\|a\|} \dashrightarrow L_{\alpha_{\|a\|}}$ is partial surjective, choose f' from $\mathcal{O}_{\|a\|}$ such that $\pi_{\|a\|}(f') = f$. Then since we have the identity $D_{a'} = \mathcal{O}_{\|a\|}$ (cf. equation (5.10)) and since Δ_a is interpreted as the graph of $\pi_{\|a\|}$ restricted to $D_{a'} \times D_a$ (cf. equation (5.13)), we have that $\Delta_a(f') = f$. So this is why the Surjectivity Axiom (3.9) comes out true on intensional structures. As for the Senses are Objects Axiom (3.10), suppose that a is a type. By definition, we have the identities $D_e = \alpha_0$ and $D_{a'} = \mathcal{O}_{\|a\|}$ (cf. equation (5.10)), and by the definition of an intensional position we have that $\mathcal{O}_{\|a\|} \subseteq \alpha_0$ (cf. part (iv) of (5.8)). Hence, on intensional structures, it is indeed the case that every sense or intension is identical to an object.

The Propositions as Fine-Grained as Objects Axiom (3.11) requires the following definition:

(5.16) An intensional position $\mathfrak{p} = (\alpha, \ell, \iota, \mathcal{O}, \pi, \nu)$ is *expressive* if there is an injection $\chi : \alpha_0 \to \pi^{-1}(\{0, 1\})$ whose graph is an element of L_α. An intensional hierarchy is *expressive* if each position in it is expressive.

It turns out that the canonical examples of intensional hierarchy are expressive (cf. [88], theorem on the classical results on the first non-computable ordinal, and existence theorem). Further, it's easy to see that expressive intensional structures model the Propositions as Fine-Grained as Objects Axiom (3.11). In particular, take the injection $\chi_1 : \alpha_0 \to \pi_1^{-1}(\{0, 1\})$. Since $D_e = \alpha_0$ and $D_t = \{0, 1\}$ and $D_{t'} = \mathcal{O}_1$ (cf. equation (5.10)), it follows that $\pi_1^{-1}(\{0, 1\}) \subseteq \mathcal{O}_1 = D_{t'}$, so that $\chi_1 : D_e \to D_{t'}$ is an injection. Moreover, note that $\Delta_t(\chi_1(x))$ is defined for each x from D_e, since $\chi_1(x) \in \pi_1^{-1}(\{0, 1\})$ and so $\pi_{\|t\|}(\chi_1(x)) = \Delta_t(\chi_1(x))$ is defined. So, as this makes clear, expressive intensional structures model the Propositions as Fine-Grained as Objects Axiom (3.11) in an interesting way since we may inject objects into propositions that actually succeed in presenting truth-values.

To finish the proof of the Predicative Consistency Theorem (4.12), it remains to establish that intensional structures are indeed models of the predicative versions of comprehension:

(5.17) (Theorem on Consistency of Predicative Comprehension) For every intensional hierarchy D (5.9), the associated intensional structure $\mathbb{D}$ models each instance of the Predicative Typed Choice Schema (4.8) and hence each instance of the Predicative Typed Comprehension Schema (4.6).

The proof of this is completed in Appendix 2 §10, since it is comparatively technical in nature. But with this, the proof of the Theorem on the Consistency of Predicative Comprehension (5.17) is finished.

6   Church's Other Axiom and Gallin's Intensional Logic

Church included another axiom in his own formulation which we have omitted in our original description of his intensional logic in §2. The strongest version of this axiom is the following, where recall that the "downarrow" notation $\downarrow$ indicates that the partial function is defined on that value (cf. discussion immediately after (2.1) in §2):

(6.1) *Iterative Axiom*: $\forall\ f', g'\ [\Delta_{ab}(f')\downarrow \neq \Delta_{ab}(g')\downarrow] \rightarrow$

$$[\exists\ x', x\ \Delta_a(x') = x\ \&\ \Delta_b(f'\langle x'\rangle) \neq \Delta_b(g'\langle x'\rangle)]$$

The motivation for this axiom is less obvious and Church said less explicitly on this subject. In my view, the best way to conceive of the motivation is as being expressive of a priority of lower-order senses over higher-order senses. The idea is that a canonical way to discern a difference between the *presentations* of higher-order senses is via a difference at the level of the *presensations* of propositions. So one knows that the sense of "wise" presents a different concept than the sense of "courageous" in part because one knows that the sense of "Zeno is wise" presents the true while the sense of "Zeno has courage" presents the false.

However, in Church's papers, this axiom was rather expressed contrapositively as follows (cf. Church's Axiom 16 [14] p. 19, [49] pp. 108-109, [1] pp. 219, 224 ff):

(6.2) $[(\forall\ x, x'(\Delta_a(x') = x \Rightarrow \Delta_b(f'\langle x'\rangle) = f(x)))\ \&\ \Delta_{ab}(f')\downarrow] \Longrightarrow \Delta_{ab}(f') = f$

In the presence of the Surjectivity Axiom (3.9) and the other axioms of Church's core system (2.8), this version follows deductively from the Iterative Axiom (6.1). To show this, suppose that the antecedent of (6.2) holds but the consequent fails, so that $\Delta_{ab}(f')\downarrow \neq f$. By the Surjectivity Axiom (3.9), choose g' of type $(ab)'$ such that $\Delta_{ab}(g') = f$. Then $\Delta_{ab}(f')\downarrow \neq \Delta_{ab}(g')\downarrow$. Then by the Iterative Axiom (6.1), we have that there is x', x such that $\Delta_a(x') = x$ and $\Delta_b(f'\langle x'\rangle) \neq \Delta_b(g'\langle x'\rangle)$. By the Typed Composition Axiom (2.5), we then have that

(6.3) $\Delta_b(f'\langle x'\rangle) \neq \Delta_b(g'\langle x'\rangle) = (\Delta_{ab}(g'))(\Delta_a(x')) = f(x)$

which contradicts the hypothesis that the antecedent of (6.2) is satisfied. So this is the sense in which the Iterative Axiom (6.1) generalizes Church's own rendition of this axiom in (6.2).

It turns out that the Iterative Axiom (6.1) (and thus also (6.2)) are true on the models that we have constructed in the previous section. This is not an accident but rather follows from some additional resources that one has available in these models, resources which allow one to interpret a fragment of Gallin's intensional logic. In particular, let us expand Church's core system (2.8) with a new function symbol ∇_a for each type a, called the *representation function*, which takes entities of type a and returns an entity of type a'. Intuitively, the idea is that the representation ∇_a function takes an extension f of type a and returns an intension f' of type a' which presents f. More formally we have the following axiom:

(6.4) *Representation Axiom*: For each entity f of type a, one has that $\nabla_a(f)$ is an entity of type a' such that $\Delta_a(\nabla_a(f)) = f$.

So note that it follows from this that the representation function ∇_a is an injection from entities of type a' to entities of type a. For, suppose that $\nabla_a(f) = \nabla_a(g)$. Then by applying the presentation function to each side and by applying the Representation Axiom (6.4) one has that

$$(6.5) \qquad f = \Delta_a(\nabla_a(f)) = \Delta_a(\nabla_a(g)) = g$$

Finally, it's perhaps also worth explicitly mentioning that the Representation Axiom (6.4) formally implies the Surjectivity Axiom (3.9). For the Surjectivity Axiom (3.9) says that each entity f of type a is presented by some intension f' of type a', and the representation function actually says that one can select the intension f' to be equal to the representation $\nabla_a(f)$.

As one can easily check, the models which we have constructed in the previous section also admit a natural interpretation of the representation function on which the Representation Axiom (6.4) comes out true. In particular, given an intensional hierarchy D (5.9), we may interpret the representation function ∇_a as the injection $\iota_{\|a\|}$ which comes built into the intensional hierarchy (where again $\|\cdot\|$ denotes the degree function (4.5) on types). So the Representation Axiom (6.4) comes out true on intensional structures simply because an intensional hierarchy was built around the idea that interpretation of the representation function is a (right) inverse to the interpretation of the presentation functions (cf. clause (v) in the definition of an intensional position (5.8)). Further, as we verify in Appendix 2 §10, the resulting intensional structure continues to model the predicative comprehension schemata (4.6)-(4.8).

The representation functions are relevant to Church's Iterative Axiom due to another axiom which holds true on the models from the last section, namely:

(6.6) *Characterization of Intensional Application*: $f'\langle x' \rangle = \nabla_b((\Delta_{ab}(f')(\Delta_a(x'))))$
One can easily check that this axiom comes out true on intensional structures by glancing at how intensional application was defined in equation (5.14). So this axiom just brings into the object language what was implicit in our constructions in the previous section. Intuitively, what this axiom is saying is that intensional application of a functional intension to an intension simply consists in figuring out what extension is presented by each, performing extensional application on these referents, and then going back to an intension via the representation function. So given this axiom, intensional application can be defined in terms of extensional application and the representation and presentation functions.

Before turning to the connection between the representation function and Church's Iterative Axiom (6.1), let's note that the axioms governing the representation function allow us to interpret a fragment of Gallin's intensional logic *IL* (cf. [39] Chapter 1). Gallin's work can be seen as an attempt to axiomatize Montague grammar ([24] Chapters 6-8, [40] Chapter 6). Montague grammar in turn can be viewed as an attempt to develop a logic motivated by possible worlds semantics in which one can distinguish between the intension and the extension of a given expression, while at the same time not having to actually quantify over

possible worlds in the object-language (cf. [39] p. 58, [24] p. 161). To this end, Montague articulated a type system– now familiar to us– in which for any type a there was a type sa which in the standard model theory is interpreted as functions from worlds to entities of type a. Montague then postulated that for every well-formed expression f of type a there is an intension $\widehat{f}$ of type sa and for every well-formed expression f' of type sa there is an extension $\widecheck{f}'$ of type a.

Gallin's later axiomatization can be seen as an attempt to see what is true on all models described by Montague. Some of the crucial axioms that Gallin set out in this intensional logic were the following (cf. [39] p. 19):

(6.7) *Axiom A2*: $\forall\, x, y\ [\widehat{x} = \widehat{y} \Longrightarrow \widehat{f(x)} = \widehat{f(y)}]$

(6.8) *Axiom A3*: $[\forall\, x\ (\widehat{f(x)} = \widehat{g(x)})] \Longrightarrow \widehat{f} = \widehat{g}]$

(6.9) *Axiom AS6*: $\widecheck{\widehat{f}} = f$

If we interpret the sa as a' and we interpret $\widehat{f}$ by $\nabla_a(f)$ when f is of type a and we interpret $\widecheck{f}'$ as $\Delta_a(f')$ when f' is of type a', then we can easily deduce these three axioms. Hence, the system of Church's intensional logic expanded with the resources of the representation function interprets a fragment of Gallin's intensional logic. One can see this observation as a partial converse to Kaplan's aforementioned possible-worlds model of Church's core system (cf. circa (2.9)). The other axioms of Gallin's intensional logic concern modal notions and lambda-terms. The system developed in this paper will have little to say about them because on the one hand it is not a modal system, and on the other hand lambda-terms, as is well known, have the force of effecting the satisfaction of the full Typed Comprehension Schema (4.1), which we do not have in our predicative setting.

In addition to their interest in providing for this interpretation of a fragment of Gallin's intensional logic, the axioms pertaining to the representation function are of interest because they deductively entail Church's Iterative Axiom.[18] But in the literature on Church's intensional logic, the Iterative Axiom (6.1) and the associated principle (6.2) have been criticized by Parsons and Klement ([64] p. 517, [51] pp. 165-166). As Anderson later put it, the

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[18]To see this, suppose that the antecedent of the Iterative Axiom (6.1) held, so that $\Delta_{ab}(f')$ was defined and not equal to $\Delta_{ab}(g')$ Then let $f = \Delta_{ab}(f')$ and let $g = \Delta_{ab}(g')$. Since f, g are functional entities of type ab, it must be the case that they differ on some value (cf. (2.6)), so that there is an object x of type a such that $f(x) \neq g(x)$. Now consider the representation $x' = \nabla_a(x)$ of this entity x. Then by the Representation Axiom (6.4), we have that x' presents x, or that $\Delta_a(x') = x$. Now, using the Characterization of Intensional Application (6.6), let us quickly compute $f'\langle x'\rangle$ and $g'\langle x'\rangle$:

(6.10) $\qquad\qquad\qquad f'\langle x'\rangle = \nabla_b((\Delta_{ab}(f')(\Delta_a(x'))) = \nabla_b(f(x))$

(6.11) $\qquad\qquad\qquad g'\langle x'\rangle = \nabla_b((\Delta_{ab}(g')(\Delta_a(x'))) = \nabla_b(g(x))$

Now, to finish the verification of the Iterative Axiom (6.1), suppose for the sake of contradiction that $f'\langle x'\rangle = g'\langle x'\rangle$. Then by the previous calculations, we see that $\nabla_b(f(x)) = \nabla_b(g(x))$. Since the representation function is injective, it then follows that $f(x) = g(x)$, contrary to hypothesis. Hence, this is why the axioms pertaining to the representation function deductively imply the Iterative Axiom (6.1).

general concern is that the Iterative Axiom (6.1) is "really quite at odds with the heuristic ideas" of Church's intensional logic, namely the formalization of fine-grained meanings ([5] p. 161). One way to see the nature of this concern is to adopt the richer perspective where we have access to the representation function. For, the axioms governing the representation function have the following consequence:

(6.12) *Characterization of Intensional Injectivity*: A function f of type ab is injective if and only if for any f' of type $(ab)'$ such that $\Delta_{ab}(f') = f$, one has that $f'\langle x'\rangle = f'\langle y'\rangle$ implies $\Delta_a(x') = \Delta_a(y')$.

The proof of this characterization is comparatively straightforward and so we relegate it to a footnote.[19] The Parsons-Klement concern can be expressed thusly: this characterization grates against some natural intuitions that one might have about fine-grained meanings.[20] Indeed, take any non-injective function that might occur naturally in language, like "the father of." If we take $a = e$ and $b = e$ and think of all the objects as consisting of persons, then this is a non-injective function f of type ab. Let's further assume for the sake of concreteness that intensions of persons are definite descriptions of some kind: THE Φ, THE Ψ, etc. Then the Characterization of Intensional Injectivity (6.12) implies that there's some sense THE FATHER OF which presents the father-of function and which is such that the

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[19]So first suppose that f is injective and that f' presents f and that we have the identity $f'\langle x'\rangle = f'\langle y'\rangle$. By the Characterization of Intensional Application (6.6), we then have the identity

(6.13) $$\nabla_b((\Delta_{ab}(f'))(\Delta_a(x'))) = \nabla_b((\Delta_{ab}(f'))(\Delta_a(y')))$$

But since the representation function ∇_b is an injection and since f' presents f, this reduces to the identity $f(\Delta_a(x')) = f(\Delta_a(y'))$ and since f is an injection, we have $\Delta_a(x') = \Delta_a(y')$, which is what we wanted to show. This completes the verification of the left-to-right direction of (6.12).

For the right-to-left direction of (6.12), suppose that f satisfies the right-hand side of (6.12). Suppose for the sake of contradiction that f is not an injection, so that $f(x) = f(y)$ but $x \neq y$. Then let $x' = \nabla_a(x)$ and $y' = \nabla_a(y)$ and $f' = \nabla_{ab}(f)$, so that the Representation Axiom (6.4) implies that x' presents x and y' presents y and f' presents f. Then we can expand the identity $f(x) = f(y)$ to

(6.14) $$(\Delta_{ab}f')(\Delta_a(x')) = f(x) = f(y) = (\Delta_{ab}f')(\Delta_a(y'))$$

Then by applying the representation function ∇_b to each side and appealing to the Characterization of the Intensional Application (6.6), we have that $f'\langle x'\rangle = f'\langle y'\rangle$. Then by the hypothesis that f' satisfies the right-hand side of (6.12), we have that $\Delta_a(x') = \Delta_a(y')$, and since x' presents x and y' presents y, we have that $x = y$, which contradicts the reductio assumption that $x \neq y$. So this completes the argument that the Characterization of Intensional Injectivity (6.12) follows from our axioms governing the representation function.

[20]See in particular Klement [51] pp. 165-166. But this is only one aspect of the concern of Parsons and Klement. First, Parsons was most interested in the interaction of Church's other axiom (6.2) with senses which do not present any referent ([64] p. 517). Second, Klement was also concerned with unintuitive consequences of axiom (6.2) related to the *intent*ionality of senses (cf. [51] p. 164). A deeper question raised by the work of Parsons and Klement is what analogues there are of the Typed Comprehension Schema (4.1) for type $(ab)'$. This is because Parsons and Klement's examples are engendered by combining senses or intensions of type $(ab)'$ together in various ways, a procedure which would be most naturally warranted by a version of the comprehension schema for senses or intensions of type $(ab)'$. I do not pretend to have addressed any of these other issues raised by the work of Parsons and Klement in this section.

intension THE FATHER OF ⟨THE Φ⟩ is the same qua intension as THE FATHER OF ⟨THE Ψ⟩, despite the fact that the person who is the Φ is not the same as the person who is the Ψ.

But consequences like this seem highly counterintuitive: one might rather have thought that if the THE FATHER OF ⟨THE BEST XYLOPHONE PLAYER⟩ is the same qua intension as THE FATHER OF ⟨THE BEST YAZHENG PLAYER⟩, then the best xylophone player is the best yazheng player. (The "$x$ylophone" and "$y$azheng" are two musical instruments which start with the same letters that are used as the variables in (6.12)). However, what we now see is that if one accepts the Characterization of Intensional Application (6.6), then one must to accept consequences like this. For, on this characterization of intensional application, the intension associated to the THE FATHER OF ⟨THE BEST XYLOPHONE PLAYER⟩ is *not* the definite description which we normally associate to the linguistic expression "the father of the best xylophone player." Rather, on this characterization of intensional application, the intension THE FATHER OF ⟨THE BEST XYLOPHONE PLAYER⟩ is the result of *intensionally* applying the intensional functional THE FATHER OF to the input of the intension THE BEST XYLOPHONE PLAYER.

Indeed, on the conception following from the Characterization of Intensional Application (6.6), this is done by first by figuring out who the father of the best xylophone player actually is– perhaps its Ted– and going and figuring out what the representation of Ted is – perhaps it is THE MAYOR OF MONTREAL. Now Ted might have two children, Alice and Bob, and it might turn out that Alice is best xylophone player while Bob is the best yazheng player. So on this conception, the intension associated to the THE FATHER OF ⟨THE BEST XYLOPHONE PLAYER⟩ is identical to the intension associated to THE FATHER OF ⟨THE BEST YAZHENG PLAYER⟩ since both are identical to the intension THE MAYOR OF MONTREAL. But in spite of this identity, the person presented by the intension THE BEST XYLOPHONE PLAYER is Alice, who is distinct from her sibling Bob, who is presented by the intension THE BEST YAZHENG PLAYER.

So while these observations show us that the characterization of intensional application (6.6) explains why intensions of functions are only as injective as the functions themselves, nonetheless it at the same time shows us that the acceptance of this characterization requires us to depart from some of the original ambitions of a fine-grained theory of intensions, on which injectivity would presumably be the rule rather than exception.[21] It is not presently obvious to us whether there is a proof of the Predicative Consistency Theorem (4.12) which would produce models which do not validate either the Characterization of Intensional Injectivity (6.12) or the Iterative Axiom (6.1). If one rejects Church's Axiom of Type Reduction (2.7) then the most difficult part of any construction of a model of these systems is to provide an interpretation of the intensional application function $(f', x') \mapsto f'\langle x' \rangle$. All the constructions which we have come up with so far have involved the aforementioned characterization of intensional application (6.6) and hence the Iterative Axiom (6.1).

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[21]For instance, some of the systems of Church and Anderson explicitly included axioms for the injectivity of senses of functional expressions. See the axiom designated "64" in Church [15] p. 151 and Anderson [1] p. 222.

7 Wehmeier and the Problem of Many Non-Extensions

In this paper, we've been primarily concerned with describing the predicative response to the Russell-Myhill paradox. However, as we've seen, predicativity constraints block the normal proof of the type-theoretic version of Cantor's Theorem (3.3). Since this theorem is closely related to Russell's paradox that there is no set of all sets, it's natural to think that there's a connection between the predicative response to the Russell-Myhill paradox of propositions and consistent fragments of the so-called naive conception of set, in which one seeks to consistently maintain that every property determines or forms a set.

To see this connection, let's note the precise way in which we can use axioms introduced thus far to produce violations of the type-theoretic version of Cantor's Theorem (3.3). In particular, let's note why these axioms give us reason to endorse the following principle:

(7.1) There is an injection ∂ from entities of type et to entities of type e such that for all f of type et there is f' of type $(et)'$ satisfying $\Delta_{et}(f') = f$ and $f' = \partial(f)$.

One proof of this proceeds via the representation function ∇_{et} introduced in the previous section (cf. circa equation (6.4)). For, one can use the Predicative Typed Comprehension Schema (4.6) and the Senses are Objects Axiom (3.10) to define the map ∂ as follows, where f is a variable of type et, x is a variable of type e, and f' is a variable of type $(et)'$:

$$(7.2) \qquad \partial(f) = x \Longleftrightarrow \exists\, f'\, (f' = \nabla_{et}(f)\ \&\ f' = x)$$

While the representation function ∇_{et} is a function from entities of type et to entities of type $(et)'$, the ∂ function is a function from entities of type et to entities of type e. Hence ∂ is an injection since as we noted in the last section the representation function ∇_{et} is an injection.

A second proof of (7.1) proceeds by recourse to the Predicative Typed Choice Schema (4.8). For, by the Surjectivity Axiom (3.9) and the Senses are Objects Axiom (3.10), one has the following, where again f is a variable of type et, x is a variable of type e, and f' is a variable of type $(et)'$:

$$(7.3) \qquad \forall\, f\, \exists\, x\, [\exists\, f'\, \Delta_{et}(f') = f\ \&\ f' = x]$$

Then by the Predicative Typed Choice Schema (4.8), it follows that there is a function ∂ of type $(et)e$ such that

$$(7.4) \qquad \forall\, f\, [\exists\, f'\, \Delta_{et}(f') = f\ \&\ f' = \partial(f)]$$

Then we may argue that ∂ is an injection. For suppose that $\partial(f) = \partial(g)$. Then by the previous equation, there are f', g' such that $f' = \partial(f) = \partial(g) = g'$ and $\Delta_{et}(f') = f$ and $\Delta_{et}(g') = g$. Then since $f' = g'$, we have that $f = \Delta_{et}(f') = \Delta_{et}(g') = g$, so that the injectivity of ∂ is thereby established.[22]

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[22]There's a obviously a deep similarity between this argument and the argument for the derivation of the contradiction in the formalized version of the Russell-Myhill paradox (3.8) in §3. The difference is that in §3 we produced an injection from entities of type $t't$ to entities of type t', while here we produce an injection from entities of type et to entities of type e.

One of the most traditional versions of the naive conception of set is that found in Frege's *Grundgesetze* ([36], [38]). One of the crucial axioms of this system is Basic Law V, which postulates the existence of a injection from concepts to objects, which Frege called the *extension operator*. Now, concepts can be identified with functions of type et (as we have had numerous occasions to observe in this paper, e.g. circa equation (3.2)). So what the previous paragraphs then shows is that the expansions of Church's intensional logic that we have studied in this paper afford the resources to satisfy one key postulate of Frege's *Grundgesetze*– namely Basic Law V– along with fragments of the comprehension schema, such as the Predicative Typed Comprehension Schema (4.6). This consistency result in and of itself is not new: versions of it were established by Parsons [63], Heck [42], and Ferreira-Wehmeier [34], and it was the focus of parts of our earlier papers [87], [88]. However, the argument of the previous paragraphs is new in that it establishes the existence of a particular *species* of extension operator, which we might dub a *sense-selecting extension operator* and define formally as follows:

(7.5) A *sense-selecting extension operator* is an injection ∂ from entities of type et to entities of type e such that for all f of type et there is f' of type $(et)'$ satisfying $\Delta_{et}(f') = f$ and $f' = \partial(f)$.

So part of what is added by looking at Frege's naive conception of set as embedded within a certain expansion of Church's intensional logic is that we have access to a particular kind of extension operator, one on which the extension of a concept is a sense of that concept.[23] Of course this general kind of maneuver is familiar from the literature on the philosophy of set theory. For instance, the stage axioms of Shoenfield [79], [80], [81] and Boolos [8] constitute an embedding of a fragment of Zermelo-Fraenkel set theory within a theory of "collections formed in stages." This gave Shoenfield and Boolos additional resources by which to respond to the Quinean charge that this set theory was just "wisdom after paradox," or just one of many ad-hoc responses to the paradoxes ([72] p. 403, cf. [71] p. 789, [68] pp. 353-354, [70] p. 5, [55] pp. 111-112).

Similarly, viewing Frege's set theory in the light of Church's intensional logic allows us to respond to a serious objection, due to Wehmeier, with these consistent fragments of the *Grundgesetze*. One way to see one's way towards this objection is to observe that working within the framework of Church's intensional logic, we can show that sense-selecting

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[23]However, it should be emphasized that the predicative response, as we have described it above in §4, is not necessarily committed to the existence of a sense-selecting extension operator. For, the two proofs from the above paragraphs used the representation operator ∇_a from §6 and the Predicative Typed Choice Schema (4.8). As stressed in §4, the philosophical motivations for the Predicative Typed Comprehension Schema (4.7) don't necessarily extend to the Predicative Typed Choice Schema (4.8); and it goes without saying that while the representation operator helps bring more of the model construction into the object-language, it too is not necessarily built into the predicative response to the Russell-Myhill paradox. Indeed, it is not even clear to me whether one can derive the existence of sense-selecting extension operators merely from the core of Church's system (2.8), the Surjectivity Axiom (3.9), the Senses are Objects Axiom (3.10), the Propositions as Fine-Grained as Objects Axiom (3.11), and the Predicative Typed Comprehension Schema (4.7). So the results of this section are only available to certain natural expansions of the predicative perspective by choice principles or by a representation operator.

extension operators have ranges which are indefinitely extensible in the sense of Dummett. To this end, let us first define some preliminary subset notation:

(7.6) *Subset Notation*: If $\Phi(x)$ is a formula in one free variable x of type a and f is an entity of type at, let's say that $h \subseteq \Phi$ if $\forall\, x\, (h(x) = 1 \to \Phi(x))$. Likewise, if g is also of type at, let's say that $h \subseteq g$ iff $\forall\, x\, (h(x) = 1 \to g(x) = 1)$, and let us define $h \subsetneq g$ as $h \subseteq g \wedge \neg(g \subseteq h)$.

Then we may define indefinite extensibility as follows:

(7.7) A formula $\Phi(x)$ in one free variable x of type a is *formally indefinitely extensible* if for each h of type at with $h \subseteq \Phi$ there is $\widetilde{h}$ of type at such that $h \subsetneq \widetilde{h} \subseteq \Phi$.

This is one formalization of the idea that "[a]n indefinitely extensible concept is one such that, if we can form a definite conception of a totality all of whose members fall under that concept, we can, by reference to that totality, characterize a larger totality of all whose members fall under it" ([30] p. 22, [25] pp. 149-150, [26] pp. 195-196, [27] p. 533, [28] p. 316, cf. [74] p. 36, [75] p. 144). If one reads Dummett's use of "concept" as any formula $\Phi(x)$ with a free object variable x and if one reads his "definite concept" as an entity of type et, then there is a comparatively tight match between the formalization in (7.7) and Dummett's own formulation of indefinite extensibility.[24]

Let's now show that if ∂ is a sense-selecting extension operator (7.5) then the range $\mathrm{rng}(\partial)$ of this extension operator is formally indefinitely extensible (7.7), where of course the range $\mathrm{rng}(\partial)$ is the following formula with x a variable of type e and f a variable of type et:

$$(7.8) \qquad\qquad (\mathrm{rng}(\partial))(x) \equiv \exists\, f\; \partial(f) = x$$

So let h be of type et such that $h \subseteq \mathrm{rng}(\partial)$, and choose a parameter q of type et. Then consider the following formula $\theta(x, f, h, q)$ where x is a variable of type e and f is a variable of type et and f' is a variable of type $(et)'$:

$$(7.9) \qquad \theta(x, f, h, q) \equiv (h(x) = 0 \wedge f = q) \vee (h(x) = 1 \wedge \exists\, f'\, \Delta_{et}(f') = f\; \&\; f' = x)$$

Let's note that for each x of type e there is unique f of type et such that $\theta(x, f, h, q)$. If $h(x) = 0$, this is trivial since the unique witness f is just the parameter q. So suppose $h(x) = 1$, so that x is in the range of the sense-selecting extension operator ∂. Then since ∂ is sense selecting, there's at least one f such that $\theta(x, f, h, q)$. Now, for uniqueness in the case that $h(x) = 1$ suppose that we have $\theta(x, f_0, h, q)$ and $\theta(x, f_1, h, q)$ with witnesses f'_0, f'_1

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[24]That said, there are some salient differences. First, this formalization provides no insight into how, if at all, $\widetilde{h}$ is provided "by reference" to h. Second, on our explication of "definite", it will follow that the definite concepts are closed under boolean operations such as intersection, union, and complement. If one has the intuition that "definite concepts" should be small in some sense, one will resist the claim that definite concepts are closed under complementation. Finally, it should be noted that this general variety of formalization of indefinite extensibility is of course not new: see for instance Shapiro-Wright [78] p. 266 and Priest [66] pp. 1264-1265.

such that $f_0' = x = f_1'$ and $\Delta_{et}(f_0') = f_0$ and $\Delta_{et}(f_1') = f_1$. Appealing tacitly to Typed Sense Determines Reference (2.4) in the form of the functionality of the presentation relations, we then have that

$$(7.10) \qquad f_0 = \Delta_{et}(f_0') = \Delta_{et}(f_1') = f_1$$

Hence, since the parameters h, q in θ have degree 2, by the Predicative Typed Comprehension Schema (4.6), we have that there is γ_h of type $e(et)$ such that $\theta(x, \gamma_h(x), h, q)$ for every x of type e.

Further, it's easy to verify the following property of γ_h:

(7.11) For every g of type et such that $h(\partial(g)) = 1$, it is the case that $\gamma_h(\partial(g)) = g$.

With all this in place,[25] we can complete the proof that $\mathrm{rng}(\partial)$ is formally indefinitely extensible. By the definition of degree in equation (4.5), note that γ_h has degree 2 since it of type $e(et)$. Hence the following formula $\varphi(x, h, \gamma_h)$, where x is a variable of type e, contains only parameters of degree 2:

$$(7.13) \qquad \varphi(x, h, \gamma_h) \equiv (h(x) = 1 \,\&\, (\gamma_h(x))(x) = 0)$$

Then by the Predicative Concept Comprehension Schema (4.7), there is g_h of type et such that

$$(7.14) \qquad g_h(x) = 1 \iff (h(x) = 1 \,\&\, (\gamma_h(x))(x) = 0)$$

Then we claim that $h(\partial(g_h)) = 0$. For, suppose not. Then let $y = \partial(g_h)$ so that $h(y) = 1$. Then by the earlier result (7.11) we have that $\gamma_h(y) = \gamma_h(\partial(g_h)) = g_h$. Then the above equation implies that

$$(7.15) \qquad g_h(y) = 1 \iff (\gamma_h(y))(y) = 0 \iff g_h(y) = 0$$

which is a contradiction. Hence indeed we have $h(\partial(g_h)) = 0$. Now, let $p = \partial(g_h)$, so that p is a parameter of type e with degree 1. Then consider the following formula which has only parameters of degree ≤ 2:

$$(7.16) \qquad \psi(x, h, p) \equiv (h(x) = 1 \lor x = p)$$

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[25]Here is the verification of (7.11). Suppose that g was of type et and satisfied $h(\partial(g)) = 1$. Let $x = \partial(g)$, so that $h(x) = 1$. Then by definition of ∂ being sense-selecting (7.5), we have $\Delta_{et}(g_0') = g$ and $g_0' = x$ for some g_0' of type $(et)'$. Then, by definition of γ_h in (7.9) and the fact that $h(x) = 1$, we have $\Delta_{et}(g_1') = \gamma_h(x)$ and $g_1' = x$ for some g_1' of type $(et)'$. Then we have $g_0' = x = g_1'$ and hence again appealing tacitly to Typed Sense Determines Reference (2.4) we may collect together the following identities

$$(7.12) \qquad g = \Delta_{et}(g_0') = \Delta_{et}(g_1') = \gamma_h(x) = \gamma_h(\partial(g))$$

which finishes the verification of (7.11).

Then by the Predicative Concept Comprehension Schema (4.7), there is $\widetilde{h}$ of type et such that

(7.17) $$\widetilde{h}(x) = 1 \Longleftrightarrow (h(x) = 1 \vee x = \partial(g_h))$$

so that in terms of our subset notation (7.6), we have $h \subsetneq \widetilde{h} \subseteq \mathrm{rng}(\partial)$ which completes the verification that the range $\mathrm{rng}(\partial)$ of a sense-selecting extension operator is formally indefinitely extensible (7.7).

Following Frege, let's call the objects falling within the range $\mathrm{rng}(\partial)$ of an extension operator ∂ the *extensions*. In this terminology, Wehmeier's observation about his consistent fragments of the *Grundgesetze* was that they required that there were infinitely many non-extensions (cf. [90] §4.2 pp. 326 ff, [91] §3 pp. 255 ff). Given the above discussion, one can see now that this follows deductively from the formally indefinite extensibility of the range $\mathrm{rng}(\partial)$ of an extension operator ∂. For, suppose that there were only finitely many objects which were non-extensions, enumerated as $q_1, \ldots, q_k$. Then consider the formula $\theta(x, q_1, \ldots, q_k)$ with free variable x of type e and parameters $q_1, \ldots, q_k$ of type e and hence degree ≤ 1:

(7.18) $$\theta(x, q_1, \ldots, q_k) \equiv (x \neq q_1 \vee \cdots \vee x \neq q_k)$$

Then by the Predicative Concept Comprehension Schema (4.7), there is h of type et such that

(7.19) $$h(x) = 1 \Longleftrightarrow (x \neq q_1 \vee \cdots \vee x \neq q_k)$$

But by hypothesis, we have that h is coextensive with the range $\mathrm{rng}(\partial)$, in that $h(x) = 1$ iff x is in the range of the extension operator ∂. But then then by the formally indefinite extensibility (7.7) of the range $\mathrm{rng}(\partial)$, there is $\widetilde{h}$ of type et such that $h \subsetneq \widetilde{h} \subseteq \mathrm{rng}(\partial)$, which contradicts that $h = \mathrm{rng}(\partial)$. Hence, the supposition that there were only finitely many non-extensions must have been wrong. Hence, the formal indefinite extensibility of the range of the extension operator requires that there be infinitely many non-extensions.

So Wehmeier's observation suggests an apparent tension between these subsystems of the *Grundgesetze* and at least some renditions of Frege's logicism (cf. *ibid*). For, sometimes logicism is described as the contention that mathematical reasoning is discoverable in every domain of inquiry (cf. ([22] p. 138, cf. [21] §VII p. 496, [19] p. 229). Presumably there are domains of inquiry (such a chemistry and biology) in which there are comparatively few non-extensions (say, finitely many atoms or organisms). So in such domains of inquiry there simply isn't "space enough" for an extension operator as axiomatized by the predicative fragments of the *Grundgesetze* of the kind considered here. Besides this apparent tension, there's a more general reason to be concerned about the problem of many non-extensions. For, this problem tells us that the presence of an extension operator has consequences for the non-extensions. So it's natural to seek an explanation for this– that is, one seeks an answer to the question: what is it about the nature of the extension operator that results in it having consequences for non-extensions?

But in the case where the extension operator is a sense-selecting extension operator, it seems that there is a natural response to the problem of many non-extensions. For, if the

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extension operator, applied to a concept, is a sense of that concept, then it is natural to expect that there will be many senses which are not extensions. For, part of the explanatory power of the Fregean doctrine of sense is that any given referent can be presented in a number of different ways. Moreover, there is no reason to expect there to be any antecedently specified finite bound on the number of different ways that a referent can be presented. Hence, because a sense-selecting extension operator selects but one sense amongst many for each concept, there will inevitably be numerous objects in these models that are not extensions. Of course, this response to the problem of many non-extensions presupposes that one is thinking about the entities of type a' as entities similar to Fregean senses in the respect that any given referent can be presented in a number of different ways. Even though Church himself was motivated by the project of axiomatizing Fregean sense, obviously there is nothing written into the axioms of Church's intensional logic or the extensions thereof considered here which forces one to adopt this presupposition.

8   Conclusions

So the last two sections have illustrated some of costs and benefits of the predicative response to the Russell-Myhill paradox of propositions, at least when expanded by certain choice-like principles or by the representation operators. In the previous section §7, we've developed a response to the Wehmeier problem of many non-extensions: the solution simply is that there are many non-extensions because the extension of a concept selects one sense from the many which present a concept. But in section §6, we saw that the consistency proof for the predicative response has some features which are not in the spirit of a fine-grained theory of intensions. For, on this model, intensions of functions are only as injective as the functions they present. Perhaps there are other model constructions which would not be committed to this result, but that question is left unresolved by the work in this paper.

Likewise, for reasons of space we have been unable to compare and contrast the versions of Church's intensional logic studied here to other formalizations of Fregean sense given by authors such as Chalmers, Horty, and Moschovakis ([10], [44], [56]), or to other formal systems of fine-grained intensions due to Fox, Lappin, Parsons, and Thomason ([35], [60], [84]). Finally, while we indicated in §3 how others like Anderson, Kaplan, and Tichý produced models which yield responses to the formalized version of the Russell-Myhill paradox (3.8), we do not pretend to have done any serious appraisal of the costs and benefits of these proposals as compared with the predictive response. Rather, we have limited ourselves here to merely setting out the predicative response in a clear manner. So in addition to suggesting a motivation for the restriction on the comprehension schema, our efforts in this paper have been directed towards establishing the formal consistency of the predicative response to the Russell-Myhill paradox of propositions.

9 Appendix 1: Proof of Proposition on Location of Domains

In this brief appendix, we prove the Proposition on Location of Domains (5.11) from §5. For ease of reference, we restate it here:

(5.11) For all $n \geq 1$, both of the following hold:

> (I) for all types a with $\|a\| < n$, there is a Σ_1-formula in parameter μ_n such that D_a is the unique element of L_{α_n} which satisfies this formula, wherein μ_n is defined by $\mu_n = \langle \nu_1, \ldots, \nu_n, \alpha_0, \alpha_1, \ldots, \alpha_{n-1} \rangle$.

> (II) for all types a with $\|a\| = n$, the set D_a is a $\underset{\sim}{\Sigma}_{\ell_n}$-definable subset of L_{α_n} in parameter μ_n.

The proof is by simultaneous induction on $n \geq 1$. For $n = 1$, note that (I) holds vacuously. As for (II), first note that if a is a type with $\|a\| = 1$, then a is among the types $e, e', e'', \ldots, t, t', t'', \ldots$. Now, if $a = e$ or $a = t$, then part (I) follows trivially since $\alpha_0 < \alpha_1$ and so both α_0 and the set $\{0, 1\}$ are members of L_{α_1} and μ_1 includes the parameter α_0 by definition. Suppose the result holds for a. Since $\mathcal{O}_1$ is $\underset{\sim}{\Sigma}_{\ell_1}$-definable subset of L_{α_1} in parameter ν_1, it follows trivially that $D_{a'} = \mathcal{O}_1$ is $\underset{\sim}{\Sigma}_{\ell_1}$-definable subset of L_{α_1} in the more complex parameter μ_1. This completes the argument in the case $n = 1$.

Now suppose that the result holds for n, and we show it holds for $n+1$. For (I), suppose that a is a type with $\|a\| < n+1$, say $\|a\| = m$. Then since D_a is a definable subset of L_{α_m} by the induction hypothesis on part (II) for m, it is an element of $L_{\alpha_m + 1}$ and hence a member of $L_{\alpha_{n+1}}$. Suppose in particular that $D_a = \{x \in L_{\alpha_m} : L_{\alpha_m} \models \psi(x, \mu_m)\}$ for some formula ψ. Then we have that D_a is the unique X in $L_{\alpha_{n+1}}$ which satisfies the following condition:

(9.1) $$\forall\, x \in L_{\alpha_m}(x \in X \leftrightarrow L_{\alpha_m} \models \psi(x, \mu_m))$$

Then since $m < n+1$ and the parameter μ_{n+1} contains the parameter μ_m as well as the ordinal α_m, and since the map $\beta \mapsto L_\beta$ is Δ_1 in $L_{\alpha_{n+1}}$ (cf. [23] II.2.8 p. 70) and since the satisfaction relation is likewise Δ_1 (cf. [23] I.9.10 p. 41), this is a Σ_1-condition in $L_{\alpha_{n+1}}$ in parameter μ_{n+1}. Here we're also appealing tacitly to the fact that the Σ_1-formulas are closed under bounded quantification in models L_α which satisfy Σ_1-collection (cf. [23] Lemma I.11.6 p. 53). This completes the induction step for part (I) of of the proposition.

For the induction step for part (II), note that the types with degree $n+1$ are of the form a' or ab. Then we may do a subinduction on complexity of type. First suppose that $\|a'\| = n+1$ and suppose that the result holds for a; we show it holds for a'. Since $\mathcal{O}_{n+1}$ is $\underset{\sim}{\Sigma}_{\ell_{n+1}}$-definable subset of $L_{\alpha_{n+1}}$ in parameter ν_{n+1}, it follows trivially that $D_{a'} = \mathcal{O}_{n+1}$ is $\underset{\sim}{\Sigma}_{\ell_{n+1}}$-definable subset of $L_{\alpha_{n+1}}$ in the more complex parameter μ_{n+1}.

Second suppose that $\|ab\| = n+1$, and suppose that the result holds for a, b; we show it holds for ab. There are two subcases here. In the first subcase, suppose that $\|a\| \geq \|b\|$. Then by the definition of degree in (4.5), we have that $\|a\|, \|b\| < \|ab\|$. Then if we let $\text{fnct}(f)$

abbreviate the Σ_0-formula expressive of the graph f being functional, and fixing similar Σ_0-definitions of $X = \text{dom}(f)$ and $Y = \text{rng}(f)$, then the set $(D_b{}^{D_a}) \cap L_{\alpha_{n+1}}$ is equal to

$$(9.2) \quad \{f \in L_{\alpha_{n+1}} : \text{fnct}(f) \ \& \ \exists \, X, Y \ X = \text{dom}(f) \ \& \ X = D_a \ \& \ Y = \text{rng}(f) \ \& \ Y = D_b\}$$

Then by part (I), we have that this is a $\underaccent{\approx}{\Sigma}_1$-definable subset of $L_{\alpha_{n+1}}$ in parameter μ_{n+1}.

Now, as a second subcase, suppose the result holds for a, b and that $\|a\| < \|b\|$, so that by the definition of degree in (4.5) we have $n+1 = \|ab\| = \|b\|$. Then D_a is a member of $L_{\alpha_{n+1}}$ by part (I), while by the supposition that the result holds for b we have that D_b is a $\underaccent{\approx}{\Sigma}_{\ell_{n+1}}$-definable subset of $L_{\alpha_{n+1}}$ in parameter μ_{n+1}. Then $(D_b{}^{D_a}) \cap L_{\alpha_{n+1}}$ is also a $\underaccent{\approx}{\Sigma}_{\ell_{n+1}}$-definable subset of $L_{\alpha_{n+1}}$ in the parameter μ_{n+1}. For, we have the following definition of $(D_b{}^{D_a}) \cap L_{\alpha_{n+1}}$:

$$(9.3) \quad \{f \in L_{\alpha_{n+1}} : L_{\alpha_{n+1}} \models [\text{fnct}(f) \ \& \ \exists \, X \ X = D_a \ \& \ \forall \, x \in X \ \exists \, y \in D_b \ \langle x, y \rangle \in f]\}$$

Here we are appealing to part (I) applied to D_a since $\|a\| < n+1$ in this case. Likewise, we are appealing to the fact that the bounded quantification in the last conjunct does not move us out of the complexity class $\underaccent{\approx}{\Sigma}_{n+1}$ in models of $\underaccent{\approx}{\Sigma}_{n+1}$-collection and $\underaccent{\approx}{\Sigma}_n$-separation. This finishes the induction step for (II). With this the inductive proof of the proposition is finished.

10 Appendix 2: Verification of the Satisfaction of Predicative Comprehension

Here we prove the following theorem from §5:

(5.17) (Theorem on Consistency of Predicative Comprehension) For every intensional hierarchy D (5.9), the associated intensional structure $\mathbb{D}$ models each instance of the Predicative Typed Choice Schema (4.8) and hence each instance of the Predicative Typed Comprehension Schema (4.6).

Further, we here prove this result for the language expanded by the representation functions $\nabla_a : D_a \to D_{a'}$ introduced in §6 (cf. circa (6.4)). As a first step towards approaching the proof of this theorem, let's first note an elementary result on terms. The terms in the signature of an intensional structures consists simply of the closure of the constants 0,1 and the variables under the extensional application symbols and the representation operations, since the presentation symbols and the intensional application symbols are not total and hence formally treated as relation symbols as opposed to function symbols. The *type* of a term is defined inductively as follows: the truth-values 0, 1 have type t, the variables have the type that they are given initially, and if τ has type ab and σ has type a, then $\tau(\sigma)$ or e-app$_{ab}(\tau, \sigma)$ has type b; and if τ has type a then $\nabla_a(\tau)$ has type a'. Then we have the following elementary result:

(10.1) (Proposition that Terms do not Raise Degree). Suppose that $\tau(x_1, \ldots, x_k)$ is a term in the signature of intensional structures with all free variables displayed such that the type of each variable x_i is $\leq n$. Then the type of the term τ is $\leq n$.

The proof is by induction on the complexity of the term. Clearly this is true in the case of the truth-values and the variables. Suppose it holds for $\tau(x_1, \ldots, x_k)$ and $\sigma(x_1, \ldots, x_k)$; we must show it is the case for e-app$_{ab}(\tau, \sigma)$ and $\nabla_a(\tau)$. First consider the case of e-app$_{ab}(\tau, \sigma)$. Then τ has type ab and σ has type a, and each has type with degree $\leq n$ by the induction hypothesis. There are two cases to consider, corresponding to the two clauses in the definition of $\|ab\|$ in (4.5). First suppose that $\|a\| \geq \|b\|$. Then one has that $\|b\| \leq \|a\| \leq n$, which is what we wanted to show since the type of e-app$_{ab}(\tau, \sigma)$ is b. Second suppose that $\|a\| < \|b\|$. Then we have that $\|b\| = \|ab\| \leq n$, which is again what we wanted to show. Finally, consider the case of $\nabla_a(\tau)$. Then τ has type a, and it has degree $\leq n$ by induction hypothesis. Then $\nabla_a(\tau)$ has type a' and so $\|a'\| = \|a\| \leq n$ by the definition of degree of $\|a'\|$ in (4.5). So this is why terms do not raise degree, or why (10.1) holds.

Relatedly, as a preliminary step, let's establish the following result about the complexity of the functions on intensional structures induced by terms:

(10.2) (Proposition on Complexity of Terms) Suppose that $\tau(\overline{u}) \equiv \tau(u_1, \ldots, u_j)$ is a term with all free variables displayed where u_i has type d_i. Since terms don't raise degree (10.1), τ has type with degree d with $\|d\| \leq m = \max\{\|d_1\|, \ldots, \|d_j\|\}$. Then τ induces a function $\tau^{\mathbb{D}} : D_{d_1} \times \cdots \times D_{d_j} \to D_d$ whose graph is $\Sigma_{\ell_m}^{L_{\alpha m}}$-definable.

Clearly this is the case if the term is variable. Now for the induction step suppose that the result holds for τ and σ; we must show it holds for $\rho(\overline{u}) \equiv$ e-app$_{e_1 e_2}(\tau(\overline{u}), \sigma(\overline{u}))$. Then $\tau(\overline{u})$ has type $e_1 e_2$ and $\sigma(\overline{u})$ has type e_1. Since terms don't raise degree (10.1), it follows that $\|e_1 e_2\|, \|e_1\|$ are all less than or equal to $m = \max\{\|d_1\|, \ldots, \|d_j\|\}$, and from this we infer that $\|e_2\| \leq \|e_1 e_2\| \leq m$ as well. Then $\rho^{\mathbb{D}} : D_{d_1} \times \cdots \times D_{d_j} \to D_{e_2}$ has the following graph:

$$\{(\overline{u}, u) \in D_{d_1} \times \cdots \times D_{d_j} \times D_{e_2} \ : \ \exists \, y \in D_{e_1} \, \exists \, z \in D_{e_1 e_2}$$
(10.3)
$$\sigma^{\mathbb{D}}(\overline{u}) = y \ \& \ \tau^{\mathbb{D}}(\overline{u}) = z \ \& \ \langle y, u \rangle \in z\}$$

This is $\Sigma_{\ell_m}^{L_{\alpha m}}$-definable by the Location of Domains (5.11) since we have that $\|e_1\|, \|e_2\|, \|e_1 e_2\| \leq m$. So this finishes the proof of (10.2). For the final induction step, suppose that the result holds for τ; we must show it holds for $\nabla_a(\tau)$. Then τ has type a, and since terms don't raise degree (10.1), it follows that $\|a\| \leq m$. Then the graph of $\tau^{\mathbb{D}}$ is $\Sigma_m^{L_{\alpha m}}$-definable. Recall from the discussion in §6 (circa (6.4)) that the representation function ∇_a is interpreted on intensional structures $\mathbb{D}$ by the function $\iota_{\|a\|}$ from the definition of an intensional hierarchy (5.9). However, this was by definition $\tau^{\mathbb{D}}$ is $\Sigma_{\|a\|}^{L_{\alpha \|a\|}}$-definable (cf. clause (iii) of the definition of an intensional position (5.8)). Since $\|a\| \leq m$, we then have that the composition $\iota_{\|a\|} \circ \tau$ is clearly also $\Sigma_m^{L_{\alpha m}}$-definable. This finishes the proof of result on the complexity of terms (10.2).

Now let's think about what kinds of symbols can appear in a formula covered by the Predicative Typed Choice Schema (4.8). So suppose that the formula $\varphi(x, y, z_1, \ldots, z_k)$ is a formula with all free variables displayed and with free variable x of type a, y of type b, and in addition variable z_i has type c_i with $\|c_i\| \leq \|ab\|$ and all the bound variables in $\varphi(x, y, z_1, \ldots, z_k)$ have type c with $\|c\| < \|ab\|$. Let $\|ab\| = n + 1$. There are then

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two important cases to consider, corresponding to the split in cases in the definition of the degree $\|ab\|$ in (4.5). If $\|a\| \geq \|b\|$, then $n+1 = \|ab\| = \|a\| + 1$ and so $\|b\| \leq \|a\| \leq n$. Further, if we split the parameter variables $z_1, \ldots, z_k$ into those that have type with degree $n+1$ and those that have type with degree $\leq n$, then we can write the formula in question as:

(10.4) (First Configuration): $\varphi(x, y, v_1, \ldots, v_m, z_1, \ldots, z_k)$ is a formula with all free variables displayed and with free variable x of type a with degree $\leq n$, y of type b with degree $\leq n$, and in addition variable v_i has type a_i with degree $\leq n$ and variable z_i has type c_i with $\|c_i\| = n+1$ and all the bound variables in the formula have type c with $\|c\| \leq n$.

Alternatively, in the other case, we have $\|a\| < \|b\|$ and $n+1 = \|ab\| = \|b\|$. If we again split the parameter variables $z_1, \ldots, z_k$ into those that have type with degree $n+1$ and those that have type with degree $\leq n$, then we can write the formula in question as:

(10.5) (Second Configuration): $\varphi(x, v_1, \ldots, v_m, y, z_1, \ldots, z_k)$ is a formula with all free variables displayed and with free variable x of type a with degree $\leq n$, y of type b with degree $n+1$, and in addition variable v_i has type a_i with degree $\leq n$ and variable z_i has type c_i with $\|c_i\| = n+1$ and all the bound variables in the formula have type c with $\|c\| \leq n$.

For ease of future reference, we call these two kinds of formulas which can feature in the Predicative Typed Choice Schema (4.8) the "first configuration" and the "second configuration".

The plan in what follows is to show that the Predicative Typed Choice Schema (4.8) holds for formulas in the second configuration (10.5), and then to show it for formulas in the first configuration (10.4). This is done by showing a result connecting the satisfaction of a formula in the second configuration to a certain level of definability in the constructible hierarchy. To build up to the statement of this result, suppose that $\varphi(x, v_1, \ldots, v_m, y, z_1, \ldots, z_k)$ is in the second configuration (10.5). Then any subformula of this formula has the form

(10.6) $$\psi(x, v_1, \ldots, v_m, v_{m+1}, \ldots v_{m+m'}, y, z_1, \ldots, z_k)$$

where the variable v_i for $i > m$ has type a_i with degree $\leq n$. Let's abbreviate $\overline{v} = \langle v_1, \ldots, v_m, v_{m+1}, \ldots v_{m+m'} \rangle$ and let's abbreviate

(10.7) $$D_{\overline{a}} = D_{a_1} \times \cdots \times D_{a_{m+m'}}, \qquad D_{\overline{c}} = D_{c_1} \times \cdots \times D_{c_k}$$

Note that since $\|a\|, \|a_i\| \leq n$, it follows from the Location of Domains (5.11), we have that $D_a \times D_{\overline{a}}$ is a member of $L_{\alpha_{n+1}}$. However, since $\|b\|, \|c_i\| = n+1$, we have that $D_b \times D_{\overline{c}}$ is a $\Sigma_{\ell_{n+1}}^{L_{\alpha_{n+1}}}$-definable subset of $L_{\alpha_{n+1}}$. Having put this terminology in place, let's now show that:

(10.8) (Proposition on Complexity of Satisfaction, Second Configuration) For every intensional hierarchy D with induced intensional structure $\mathbb{D}$ and every subformula $\psi(x, \overline{v}, y, \overline{z})$ of a formula in the second configuration (10.5), the following set is $\Sigma_{\ell_{n+1}}^{L_{\alpha_{n+1}}}$-definable:

$$[\psi]^D = \{(x, \overline{v}, y, \overline{z}) \in D_a \times D_{\overline{a}} \times D_b \times D_{\overline{c}} : \mathbb{D} \models \psi(x, \overline{v}, y, \overline{z})\}$$

We establish this by induction on the complexity of the subformula. By pushing all the negations to the inside, it suffices to show that the result holds for atomics, negated atomics, and is closed under conjunctions, disjunctions, existential quantification, and universal quantification. Let's being with the atomic case, considering the negated atomic cases along the way. The atomic formulas in intensional structures have three possible forms, namely:

$$(10.9) \qquad\qquad \tau = \sigma, \qquad \Delta_d(\tau) = \sigma, \qquad \text{i-app}_{c_0 d_0}(\tau, \sigma) = \rho$$

where τ, σ, ρ are terms. These are the only possible subformulas because technically, the second is shorthand for the binary atomic formula $\Delta_d(\tau, \sigma)$ and the third is shorthand for the associated ternary atomic relation (cf. discussion circa equations (5.13) and (5.14)). Since τ, σ, ρ appear in a formula in the second configuration (10.5), the free variables in these terms τ, σ, ρ have respective types e_1, e_2, e_3 with degree $\leq n+1$ and since terms don't raise degree (10.1), it follows that $\|e_1\|, \|e_2\|, \|e_3\| \leq n+1$. From this it also follows that $\|d\| \leq n+1$ and $\|c_0\|, \|d_0\| \leq \|c_0 d_0\| = \|(c_0 d_0)'\| \leq n+1$.

Let's consider first the case of equality between terms, that is atomic formulas of the form

$$(10.10) \qquad\qquad \psi(x, \overline{v}, y, \overline{z}) \equiv \tau(x, \overline{v}, y, \overline{z}) = \sigma(x, \overline{v}, y, \overline{z})$$

Then we have that

$$(x, \overline{v}, y, \overline{z}) \in [\psi]^D \iff \exists\, z_1 \in D_{e_1}, \exists\, z_2 \in D_{e_2}\; \tau(x, \overline{v}, y, \overline{z}) = z_1$$
$$(10.11) \qquad\qquad\qquad\qquad\qquad \&\; \sigma(x, \overline{v}, y, \overline{z}) = z_2, \;\&\; z_1 = z_2$$

which is $\Sigma^{L_{\alpha_{n+1}}}_{\ell_{n+1}}$-definable by the result on the complexity of terms (10.2). Similarly we have that

$$(x, \overline{v}, y, \overline{z}) \in [\neg\psi]^D \iff \exists\, z_1 \in D_{e_1}, \exists\, z_2 \in D_{e_2}\; \tau(x, \overline{v}, y, \overline{z}) = z_1$$
$$(10.12) \qquad\qquad\qquad\qquad\qquad \&\; \sigma(x, \overline{v}, y, \overline{z}) = z_2 \;\&\; z_1 \neq z_2$$

which is $\Sigma^{L_{\alpha_{n+1}}}_{\ell_{n+1}}$-definable for the same reasons.

Now let's consider the case of the presentation symbols, that is atomic formulas of the form

$$(10.13) \qquad\qquad \psi(x, \overline{v}, y, \overline{z}) \equiv \Delta_d(\tau(x, \overline{v}, y, \overline{z})) = \sigma(x, \overline{v}, y, \overline{z})$$

Then $[\psi]^D$ is $\Sigma^{L_{\alpha_{n+1}}}_{\ell_{n+1}}$-definable since we have

$$(x, \overline{v}, y, \overline{z}) \in [\psi]^D \iff \exists\, z_1 \in D_{e_1}, \exists\, z_2 \in D_{e_2}\; \tau(x, \overline{v}, y, \overline{z}) = z_1$$
$$(10.14) \qquad\qquad\qquad\qquad\qquad \&\; \sigma(x, \overline{v}, y, \overline{z}) = z_2 \;\&\; \pi_{\|d\|}(z_1) = z_2$$

and since $\|d\| \leq n+1$ implies that $\pi_{\|d\|}$ is $\Sigma^{L_{\alpha_{n+1}}}_{\ell_{n+1}}$-definable. Further, by using the fact that $\mathcal{O}_{\|d\|} \setminus \pi^{-1}_{\|d\|}(L_{\alpha_{\|d\|}})$ is $\Sigma^{L_{\alpha_{\|d\|}}}_{\|d\|}$-definable (cf. clause (v) of the definition of an intensional

position (5.8)) and that $\|d\| \leq n + 1$, we have that

$$
\begin{aligned}
(x, \overline{v}, y, \overline{z}) \in [\neg\psi]^D \iff & \exists z_1 \in D_{e_1}, \exists z_2 \in D_{e_2}, \exists z_3 \in L_{\alpha_{n+1}} \\
& \tau(x, \overline{v}, y, \overline{z}) = z_1 \ \& \ \sigma(x, \overline{v}, y, \overline{z}) = z_2 \\
& \& \ (z_1 \in \mathcal{O}_{\|d\|} \setminus \pi_{\|d\|}^{-1}(L_{\alpha_{\|d\|}})) \vee (\pi_{\|d\|}(z_1) = z_3 \ \& \ z_2 \neq z_3)
\end{aligned}
$$
(10.15)

As the final atomic case, consider the case of intensional application:

(10.17) $\qquad \psi(x, \overline{v}, y, \overline{z}) \equiv \text{i-app}_{c_0 d_0}(\tau(x, \overline{v}, y, \overline{z}), \sigma(x, \overline{v}, y, \overline{z})) = \rho(x, \overline{v}, y, \overline{z})$

Then the type e_1 of τ must be $(c_0 d_0)'$ and the type e_2 of σ must be c_0'. Then we have

$$
\begin{aligned}
(x, \overline{v}, y, \overline{z}) \in [\psi]^D \iff & \exists z_1 \in D_{(c_0 d_0)'}, \exists z_2 \in D_{c_0'}, \exists z_3 \in D_{d_0'} \\
& \tau(x, \overline{v}, y, \overline{z}) = z_1 \ \& \ \sigma(x, \overline{v}, y, \overline{z}) = z_2 \ \& \ \rho(x, \overline{v}, y, \overline{z}) = z_3 \\
& \& \ \iota_{\|d_0\|}((\pi_{\|c_0 d_0\|}(z_1))(\pi_{\|c_0\|}(z_2))) = z_3
\end{aligned}
$$
(10.18)

Since $\|c_0\|, \|d_0\| \leq \|c_0 d_0\| = \|(c_0 d_0)'\| \leq n+1$, we have that $\pi_{\|c_0 d_0\|}, \pi_{\|c_0\|}, \iota_{\|d_0\|}$ are all $\Sigma_{\ell_{n+1}}^{L_{\alpha_{n+1}}}$-definable. Finally, for the negation, we may argue as follows, again appealing to the fact that in the definition of an intensional position we required that the set $\mathcal{O}_j \setminus \pi_j^{-1}(L_{\alpha_j})$ was $\Sigma_{\ell_j}^{L_{\alpha_j}}$-definable for all $j \geq 1$ (cf. clause (v) of the definition of an intensional position (5.8)):

$$
\begin{aligned}
(x, \overline{v}, y, \overline{z}) \in [\neg\psi]^D \iff & \exists z_1 \in D_{(c_0 d_0)'}, \exists z_2 \in D_{c_0'}, \exists z_3 \in D_{d_0'}, \exists z_4 \in L_{\alpha_{n+1}} \\
& \tau(x, \overline{v}, y, \overline{z}) = z_1 \ \& \ \sigma(x, \overline{v}, y, \overline{z}) = z_2 \ \& \ \rho(x, \overline{v}, y, \overline{z}) = z_3 \\
& \wedge [(z_1 \in \mathcal{O}_{\|c_0 d_0\|} \setminus \pi_{\|c_0 d0\|}^{-1}(L_{\alpha_{\|c_0 d_0\|}})) \\
& \vee (z_2 \in \mathcal{O}_{\|c_0\|} \setminus \pi_{\|c_0\|}^{-1}(L_{\alpha_{\|c_0\|}})) \\
& \vee (\iota_{\|d_0\|}((\pi_{\|c_0 d_0\|}(z_1))(\pi_{\|c_0\|}(z_2)) = z_4 \ \& \ z_4 \neq z_3)]
\end{aligned}
$$
(10.19)

This completes the base cases of the inductive argument for (10.8). Since $\Sigma_{\ell_{n+1}}^{L_{\alpha_{n+1}}}$-definability is closed under finite intersections and unions, the inductive steps for conjunction and disjunction are trivial. Let us then consider the case of universal quantification. Suppose that the result holds for $\psi(x, \overline{v}, v, y, \overline{z})$ and let us show it holds for $\theta(x, \overline{v}, y, \overline{z}) \equiv \forall v_0 \ \psi(x, \overline{v}, v_0, y, \overline{z})$. Since this is a subformula of a formula in the second configuration (10.5), it follows the bound variable v_0 has a type a_0 with degree $\leq n$. Then by part (ii) of the result on Locations of Domains (5.11), it follows that $X = D_{a_0}$ is a $\Sigma_1^{L_{\alpha_{n+1}}}$-condition. Then one has that

(10.21) $\qquad (x, \overline{v}, y, \overline{z}) \in [\theta]^D \iff \exists X \ X = D_{a_0} \ \& \ \forall v_0 \in X \ (x, v_0, \overline{v}, y, \overline{z}) \in [\psi]^D$

so that $[\theta]^D$ is likewise $\Sigma_{\ell_{n+1}}^{L_{\alpha_{n+1}}}$-definable since $\Sigma_{\ell_{n+1}}^{L_{\alpha_{n+1}}}$-definability is closed under bounded quantification in models $L_{\alpha_{n+1}}$ of Σ_{n+1}-collection and Σ_n-separation. A similar argument holds in the case of the existential quantifier, but is even easier since there we do not have to appeal to this result about closure under bounded quantification. So this finishes the

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result on the complexity of satisfaction in the case of a formula which is in the second configuration (10.8).

Now let us finally establish that the Predicative Typed Choice Schema (4.6) holds on intensional structures, at first with respect to formulas in the second configuration (10.5). So suppose that the antecedent holds:

(10.22) $$\mathbb{D} \models \forall\, x \, \exists\, y \; \varphi(x, y, p_1, \ldots, p_m, q_1, \ldots, q_k)$$

where $\varphi(x, y, v_1, \ldots, v_m, z_1, \ldots z_k)$ is in the second configuration (10.5). Consider the following relation:

(10.23) $$R(x, y) \equiv [x \in D_a \;\&\; y \in D_b \;\&\; \mathbb{D} \models \varphi(x, y, p_1, \ldots, p_m, q_1, \ldots, q_k)]$$

Then by the result on the complexity of satisfaction (10.8), one has that R is $\Sigma^{L_{\alpha_{n+1}}}_{\ell_{n+1}}$-definable. And by equation (10.22), one has that

(10.24) $$L_{\alpha_{n+1}} \models \forall\, x \in D_a \; \exists\, y \; R(x, y)$$

By the uniformization theorem (cf. [47] Theorem 3.1 p. 256 and Lemma 2.15 p. 255; [23] Theorem 4.5 p. 269, [88]), choose a $\Sigma^{L_{\alpha_{n+1}}}_{\ell_{n+1}}$-definable relation $R' \subseteq R$ such that

(10.25) $$L_{\alpha_{n+1}} \models [\forall\, x \; (\exists\, y \; R(x, y)) \rightarrow (\exists\,!\, y \; R'(x, y))]$$

Then by equation (10.24), one has that R' is the graph of a function $h : D_a \to D_b$. Since this graph is $\Sigma^{L_{\alpha_{n+1}}}_{\ell_{n+1}}$-definable with domain an element of $L_{\alpha_{n+1}}$, by Replacement (cf. [23] Lemma I.11.7 p. 53), one has that it is an element of $L_{\alpha_{n+1}} = L_{\alpha_{\|ab\|}}$, where a is the type of the variable x from the formula φ from equation (10.22) and b is the type of the variable y. Then h is an element of the domain $D_b^{D_a} \cap L_{\alpha_{\|ab\|}} = D_{ab}$ (cf. the third clause of equation (5.10)). Hence, we've shown that there is h in D_{ab} such that

(10.26) $$\mathbb{D} \models \forall\, x \; \varphi(x, h(x), p_1, \ldots, p_m, q_1, \ldots, q_k)$$

which is what we were required to show in the consequent of Predicative Typed Choice Schema (4.6).

So we've verified that the Predicative Typed Choice Schema (4.6) holds on intensional structures, at least with respect to formulas in the second configuration (10.5). Let's now argue that the same holds with respect to formulas in the first configuration (10.4). So suppose that

(10.27) $$\mathbb{D} \models \forall\, x \, \exists\, y \; \varphi(x, y, p_1, \ldots, p_m, q_1, \ldots, q_k)$$

where $\varphi(x, y, v_1, \ldots, v_m, z_1, \ldots z_k)$ is in the first configuration (10.4). Then consider the following, where w is a variable of type ab with degree $n + 1$ and x_1, x_2 are variables of type a:

(10.28) $$\psi(x, w, \overline{p}, \overline{q}) \equiv (\forall\, x_1, x_2 \; w(x_1) = w(x_2)) \;\&\; (\exists\, x_1, y \; w(x_1) = y) \;\&\; \varphi(x, y, \overline{p}, \overline{q})$$

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Intuitively this formula ψ is saying that w is a constant function of type ab and its constant value is a witness to φ. Now ψ is in the second configuration (10.5), and we can verify by hand that for every element y of D_b there is a constant function $w(x_1) = y$ of type D_{ab} since if $y \in L_{\alpha_{\|b\|}}$ then $\{\langle x_1, y \rangle : x_1 \in L_{\alpha_{\|a\|}}\}$ is in $L_{\alpha_{\|ab\|}}$. Then by Predicative Typed Choice Schema (4.6) applied to ψ, we have that there is an element h of type $a(ab)$ such that $\mathbb{D} \models \forall\, x\ \psi(x, h(x), \overline{p}, \overline{q})$. Note that since $\|ab\| = n + 1 = \|a(ab)\|$ we have that h is in $L_{\alpha_{n+1}}$. Then the $g : D_a \to D_b$ such that $g(x) = (h(x))(x)$ is in $L_{\alpha_{n+1}}$ by Σ_0-separation since

$$(10.29) \qquad\qquad g = \{\langle x, y \rangle \in D_a \times D_b : \langle x, \langle x, y \rangle\rangle \in h\}$$

So we've shown that g is an element of D_{ab} and by construction we have

$$(10.30) \qquad\qquad \mathbb{D} \models \forall\, x\ \varphi(x, g(x), p_1, \ldots, p_m, q_1, \ldots, q_k)$$

so that we also have that Predicative Typed Choice Schema (4.6) holds on intensional structures, regardless of which of the two configurations we are in.

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