Bisimulation Quotient in Inquisitive Modal Logic

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Abstract. Inquisitive logic is an extension of classical logic, with a new connective $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$, which can express questions. Similarly, inquisitive modal logic, InqML, is a natural generalization of standard Kripke-style modal logic with $\boxplus$ as a basic modal operator. In this paper, we observe a bisimulation quotient in inquisitive modal logic. Given model $\mathfrak{M}$, we first show that a relation of bisimilarity is an equivalence relation and that there exists the largest bisimulation on that model. Furthermore, we define the bisimulation quotient and show that two models are globally bisimilar if and only if their bisimulation quotients are isomorphic.

Keywords: Inquisitive modal logic · Bisimulation · Bisimulation quotient

1 Introduction

Inquisitive logic can be seen as a generalization of classical logic which contains not only statements, but also questions. The language of classical propositional logic is extended with a new connective $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$, called *inquisitive disjunction*. For example, $p \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} q$ can be read as "whether p or q is the case". Although it is not clear what it means for a question to be true or false, there is a sense in which question can be said to be settled by given information, so, instead of a truth-semantics, a support-semantics has been used, i.e. a semantics in which, instead of relation between worlds and formulas, a support relation is established between sets of worlds and formulas.

Inquisitive modal logic, InqML, as a generalization of standard modal logic, was first introduced in [1]. A new modal operator $\boxplus$, called *window*, has been introduced.

In standard modal logic, an accessibility function σ to each world $w \in W$ assigns a set $\sigma(w) \subseteq W$ of worlds. For a formula φ, $\Box\varphi$ expresses the fact that $\sigma(w) \subseteq |\varphi|$, where $|\varphi| \subseteq W$ is a set of worlds where the formula φ is true.

In inquisitive modal logic, a function Σ to each world $w \in W$ assigns a set of sets of worlds, i.e. $\Sigma(w) \subseteq \mathcal{P}(W)$. If $[\varphi]$ is a set of sets of worlds where the formula φ is supported, then $\boxplus\varphi$ expresses the fact that $\Sigma(w) \subseteq [\varphi]$. This shows that InqML is indeed a natural generalization of standard modal logic.

The notion of bisimulation in InqML was introduced in [2]. Ciardelli and Otto proved an Ehrenfeucht-Fraïssé theorem for InqML, i.e. n-bisimilarity coincides with n-modal equivalence. Furthermore, they characterize inquisitive modal logic as the bisimulation invariant fragment of first-order logic over various classes of two-sorted structures.

In this paper, we show that, for an arbitrary model $\mathfrak{M} = (W, \Sigma, V)$, a relation of bisimilarity is an equivalence relation on W and that there is the largest bisimulation on that model. Furthermore, we define the bisimulation quotient and show that a model is related by its bisimulation quotient by a surjective bounded morphism which implies modal equivalence between the model and its bisimulation quotient.

Finally, we prove the main result of the paper, i.e. two models are globally bisimilar if and only if their bisimulation quotients are isomorphic.

2 Preliminaries

The following definitions are taken from [1] and [2].

Let W be a non-empty set of elements called the *worlds*. An *information state* (or simply, *state*) over W is any subset $s \subseteq W$. Furthermore, an *inquisitive state* over W is a non-empty set of information states $\Pi \subseteq \mathcal{P}(W)$ that is downward closed, i.e. $s \in \Pi$ implies $t \in \Pi$ for all $t \subseteq s$.

Let P be a set of propositional variables. An *inquisitive modal frame* is a pair $F = (W, \Sigma)$, where W is a set of worlds and $\Sigma : W \to \mathcal{P}(\mathcal{P}(W))$ a function that to each world $w \in W$ assigns an inquisitive state $\Sigma(w)$. An *inquisitive modal model* is a triple $\mathfrak{M} = (W, \Sigma, V)$, where (W, Σ) is an inquisitive modal frame and $V : \mathrm{P} \to \mathcal{P}(W)$ a function that to each propositional variable assigns a set of worlds.

The syntax of inquisitive modal logic InqML is given as follows:

$$\varphi := p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \varphi \mathbin{\vee\!\!\!\vee} \varphi \mid \boxplus\varphi.$$

The semantics of InqML is given in terms of support in an information state, i.e. if $\mathfrak{M} = (W, \Sigma, V)$ is an inquisitive modal model and $s \subseteq W$, then we define:

- $\mathfrak{M}, s \models p \iff s \subseteq V(p)$;
- $\mathfrak{M}, s \models \bot \iff s = \emptyset$;
- $\mathfrak{M}, s \models \varphi \wedge \psi \iff \mathfrak{M}, s \models \varphi$ and $\mathfrak{M}, s \models \psi$;
- $\mathfrak{M}, s \models \varphi \to \psi \iff$ for all $t \subseteq s$, $\mathfrak{M}, t \models \varphi$ implies $\mathfrak{M}, t \models \psi$;
- $\mathfrak{M}, s \models \varphi \mathbin{\vee\!\!\!\vee} \psi \iff \mathfrak{M}, s \models \varphi$ or $\mathfrak{M}, s \models \psi$;
- $\mathfrak{M}, s \models \boxplus\varphi \iff$ for all $w \in s$, for all $t \in \Sigma(w)$, $\mathfrak{M}, t \models \varphi$.

We take negation and disjunction as defined connectives in the usual way:

$$\neg\varphi := \varphi \to \bot \text{ and } \varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi).$$

We say that a state s is *compatible* with a formula φ if there is $t \subseteq s, t \neq \emptyset$, such that $\mathfrak{M}, t \models \varphi$. Using this notion, the support conditions for connectives $\neg$ and $\vee$ can be expressed as follows:

- $\mathfrak{M}, s \models \neg\varphi \iff s$ is not compatible with φ;

- $\mathfrak{M}, s \models \varphi \vee \psi \iff$ for all $t \subseteq s$, $t \neq \emptyset$, we have that t is compatible with either φ or ψ.

The following properties hold generally in InqML:

- *Persistency:* if $\mathfrak{M}, s \models \varphi$ and $t \subseteq s$, then $\mathfrak{M}, t \models \varphi$;
- *Empty state property:* $\mathfrak{M}, \emptyset \models \varphi$.

Remark 1. With any inquisitive modal model $\mathfrak{M} = (W, \Sigma, V)$ we can associate Kripke model $\mathfrak{M}_K = (W, \sigma, V)$, where $\sigma : W \to \mathcal{P}(W)$ is a function defined by $\sigma(w) = \bigcup \Sigma(w)$.

Remark 2. In the literature, the modal operator $\Box$ is included in the alphabet, with the following support condition:

$$\mathfrak{M}, s \models \Box\varphi \iff \text{ for all } w \in s \text{ we have } \mathfrak{M}, \sigma(w) \models \varphi.$$

However, it is shown that $\Box$ is definable from $\boxplus$ in a way that every formula of the form $\Box\varphi$ is equivalent to disjunctions of formulas of the form $\boxplus\psi$ (cf. [1] for details), so, in this paper, we did not include $\Box$ in the alphabet.

3 Bisimilarity as an equivalence relation

In this section we define a notion of bisimulation in InqML, then we prove that the bisimilarity relation is an equivalence relation and show that there is the largest bisimulation between two models.

The following definitions are taken from [2].

The *lifting* of a relation $Y \subseteq W \times W'$ is the relation $\overline{Y} \subseteq \mathcal{P}(W) \times \mathcal{P}(W')$ defined with: $s\overline{Y}s'$ if and only if the following conditions are satisfied:

- for all $w \in s$ there exists $w' \in s'$ such that wYw';
- for all $w' \in s'$ there exists $w \in s$ such that wYw'.

Definition 1. *Let $\mathfrak{M} = (W, \Sigma, V)$ and $\mathfrak{M}' = (W', \Sigma', V')$ be inquisitive modal models. A non-empty relation $Z \subseteq W \times W'$ is called a **bisimulation** if the following conditions are satisfied:*

(at) if wZw', then $w \in V(p)$ if and only if $w' \in V'(p)$, for all $p \in \mathrm{P}$;
(forth) if wZw' and $s \in \Sigma(w)$, then there exists $s' \in \Sigma'(w')$ such that $s\overline{Z}s'$;
(back) if wZw' and $s' \in \Sigma'(w')$, then there exists $s \in \Sigma(w)$ such that $s\overline{Z}s'$.

*We say that the worlds w and w' are **bisimilar**, denoted $w \sim w'$, if there exists a bisimulation Z such that wZw'.*

*We say that the states s and s' are **bisimilar**, denoted $s \sim s'$, if there exists a bisimulation Z such that $s\overline{Z}s'$.*

Remark 3. Alternatively, a bisimulation between two inquisitive modal models $\mathfrak{M} = (W, \Sigma, V)$ and $\mathfrak{M}' = (W', \Sigma', V')$ can be defined as a non-empty relation $Z \subseteq W \times W' \cup \mathcal{P}(W) \times \mathcal{P}(W')$ (cf. [2] for details). In this paper, we chose to use the notion of bisimulation as a relation defined exclusively on the worlds of the observed models.

Remark 4. It is shown in [2] that if the states s and s' are bisimilar, then s and s' are modally equivalent, i.e. s and s' are supports for the same formulas.

Let $\mathfrak{M} = (W, \Sigma, V)$ be an arbitrary model. Consider the relation of bisimilarity between worlds of $\mathfrak{M}$, i.e. for $w, u \in W$, $w \sim u$ if and only if there is a bisimulation $Z \subseteq W \times W$ such that wZu.

Now we prove that the bisimilarity relation is an equivalence relation on W.

It is obvious that the bisimilarity relation is a reflexive and symmetric relation as we state in the next proposition.

Proposition 1. *Let $\mathfrak{M} = (W, \Sigma, V)$ be an inquisitive modal model and $Z \subseteq W \times W$ a bisimulation. Then for every $w \in W$, we have wZw. Furthermore, if wZu, then uZw.*

In the following proposition we show that the composition of bisimulations is a bisimulation, so we can conclude that the bisimilarity relation is a transitive relation on W.

Proposition 2. *Let $\mathfrak{M} = (W, \Sigma, V)$, $\mathfrak{M}' = (W', \Sigma', V')$ and $\mathfrak{M}'' = (W'', \Sigma'', V'')$ be inquisitive modal models. If $Z' \subseteq W \times W'$ and $Z'' \subseteq W' \times W''$ are bisimulations, then $Z = Z' \circ Z'' \subseteq W \times W''$ is a bisimulation.*

Proof.

(at) Let wZw''. Then there is w' such that $wZ'w'$ and $w'Z''w''$. Since Z' and Z'' are bisimulations, it follows: $w \in V(p)$ if and only if $w' \in V'(p)$ if and only if $w'' \in V''(p)$.

$(forth)$ Let wZw'' and $s \in \Sigma(w)$. Then there is w' such that $wZ'w'$ and $w'Z''w''$. Since Z' is a bisimulation, there exists $s' \in \Sigma'(w')$ such that $s\overline{Z'}s'$. Now since Z'' is a bisimulation, there exists $s'' \in \Sigma''(w'')$ such that $s'\overline{Z''}s''$. We claim that $s\overline{Z}s''$. Suppose $u \in s$. Since $s\overline{Z'}s'$, there is some $u' \in s'$ such that $uZ'u'$ and, since $s'\overline{Z''}s''$, there is some $u'' \in s''$ such that $u'Z''u''$. Hence, uZu''. The second condition can be proved similarly.

$(back)$ Similarly as the condition $(forth)$.

Hence, $Z = Z' \circ Z''$ is a bisimulation.

Hence, propositions 1 and 2 imply that the relation $\sim$ is an equivalence relation on W.

Proposition 3. *Let $\mathfrak{M} = (W, \Sigma, V)$ and $\mathfrak{M}' = (W', \Sigma', V')$ be inquisitive modal models. If $(Z_i, i \in I)$ is a family of bisimulations $Z_i \subseteq W \times W'$, then $Z = \bigcup_{i \in I} Z_i$ is a bisimulation.*

Proof.

(at) Let wZw'. Then there is some $i \in I$ such that wZ_iw'. Since Z_i is a bisimulation, it follows $w \in V(p)$ if and only if $w' \in V'(p)$.

($forth$) Let wZw' and $s \in \Sigma(w)$. Then there is some $i \in I$ such that wZ_iw'. Since Z_i is a bisimulation, there exists $s' \in \Sigma'(w')$ such that $s\overline{Z_i}s'$.
We claim that $s\overline{Z}s'$. Suppose $u \in s$. Since $s\overline{Z_i}s'$, there is some $u' \in s'$ such that uZ_iu'. That implies uZu'. The second condition can be proved similarly.

($back$) Similarly as the condition ($forth$).

Hence, $Z = \bigcup_{i \in I} Z_i$ is a bisimulation.

From the previous proposition we conclude that there exists the largest bisimulation between models $\mathfrak{M}$ and $\mathfrak{M}'$.

Let $\mathfrak{M} = (W, \Sigma, V)$ be an inquisitive modal model and Z the largest bisimulation on $\mathfrak{M}$. Then Z is an equivalence relation. For $w \in W$, we denote by $[w]$ the equivalence class of an element w. That class consists of all elements $u \in W$ such that wZu, i.e. we have wZu if and only if $[w] = [u]$.

Furthermore, given $s \subseteq W$, let

$$[s] = \{[w] : \text{there exists } u \in s \text{ such that } wZu\}.$$

As for worlds, for states we have:

$$s\overline{Z}s' \text{ if and only if } [s] = [s'].$$

To show this, first suppose $s\overline{Z}s'$. Take any $[w] \in [s]$. Then there is $u \in s$ such that wZu. Since $s\overline{Z}s'$, there exists $u' \in s'$ such that uZu', so we have wZu', i.e. $[w] = [u']$. From $u' \in s'$ it follows $[u'] \in [s']$, so we have $[w] \in [s']$. Thus, $[s] \subseteq [s']$. We can prove $[s'] \subseteq [s]$ similarly. Conversely, let $[s] = [s']$. Take any $w \in s$. Then we have $[w] \in [s]$. That implies $[w] \in [s']$. That means that there exists $u' \in s'$ such that wZu'. The second condition can be proved similarly, so we have $s\overline{Z}s'$.

4 Bisimulation quotient

In this section we define the bisimulation quotient and prove that two models are globally bisimilar if and only if they have isomorphic quotients.

Definition 2. *Let $\mathfrak{M} = (W, \Sigma, V)$ be an inquisitive modal model and Z the largest bisimulation on $\mathfrak{M}$. The model $\mathfrak{M}_Z = (W_Z, \Sigma_Z, V_Z)$ is called the* ***bisimulation quotient*** *of $\mathfrak{M}$ if the following conditions are satisfied:*

- $W_Z = \{[w] : w \in W\}$;
- $[s] \in \Sigma_Z([w])$ *iff there are* $w' \in [w]$ *and* s' *such that* $[s'] = [s]$ *and* $s' \in \Sigma(w')$;
- $[w] \in V_Z(p)$ *iff* $w \in V(p)$, *for all* $p \in \mathrm{P}$.

Naturally we expect a model to be related to its bisimulation quotient by a surjective bounded morphism, and thus by modal equivalence (cf. [3]). Here we adapt this for inquisitive modal logic.

Definition 3. *Let $\mathfrak{M} = (W, \Sigma, V)$ and $\mathfrak{M}' = (W', \Sigma', V')$ be inquisitive modal models. A function $f : \mathfrak{M} \to \mathfrak{M}'$ is called a* ***bounded morphism*** *of the models $\mathfrak{M}$ and $\mathfrak{M}'$ if the following conditions are satisfied:*

(at_{bm}) *$w \in V(p)$ if and only if $f(w) \in V'(p)$, for all $p \in \mathrm{P}$;*
$(forth_{bm})$ *if $s \in \Sigma(w)$, then $f(s) \in \Sigma'(f(w))$;*
$(back_{bm})$ *if $s' \in \Sigma'(f(w))$, then there is $s \in \Sigma(w)$ such that $f(s) = s'$.*

In the previous definition, for $s \subseteq W$, we take $f(s)$ as $f(s) = \{f(w) : w \in W\}$.

Now we show that modal satisfaction is invariant under bounded morphism, i.e. the following proposition holds.

Proposition 4. *Let $f : \mathfrak{M} \to \mathfrak{M}'$ be a bounded morphism of inquisitive modal models $\mathfrak{M} = (W, \Sigma, V)$ and $\mathfrak{M}' = (W', \Sigma', V')$. Then for every $s \subseteq W$ and every formula φ we have*

$$\mathfrak{M}, s \models \varphi \text{ if and only if } \mathfrak{M}', f(s) \models \varphi.$$

Proof. By Remark 4, it is sufficient to show that there exists a bisimulation Z such that $s\overline{Z}f(s)$. Let $Z = \{(w, f(w))) : w \in W\}$.

Suppose $wZf(w)$. Then we have $w \in V(p)$ if and only if $f(w) \in V'(p)$, so the condition (at) holds.

Suppose $wZf(w)$ and $s \in \Sigma(w)$. From the $(forth_{bm})$ condition, it follows $f(s) \in \Sigma'(f(w))$. We need to show $s\overline{Z}f(s)$. If $u \in s$, then for $f(u) \in f(s)$ we have $uZf(u)$. Conversely, if $u' \in f(s)$, then there exists $u \in s$ such that $f(u) = u'$. That implies $uZf(u)$, i.e. uZu'. Hence, the $(forth)$ condition holds.

Let $wZf(w)$ and $s' \in \Sigma'(f(w))$. From the $(back_{bm})$ condition, it follows that there exists $s \in \Sigma(w)$ such that $f(s) = s'$. As in the previous case, it can be proved that $s\overline{Z}s'$, so the $(back)$ condition also holds.

Hence, Z is a bisimulation.

Let $s \subseteq W$ be an arbitrary state. Suppose $u \in s$. Then for $f(u) \in f(s)$ we have $uZf(u)$. Conversely, suppose $u' \in f(s)$. Then there exists $u \in s$ such that $f(u) = u'$. Now we have $uZf(u)$, i.e. uZu'. Hence, $s\overline{Z}f(s)$.

Proposition 5. *The projection $\pi : \mathfrak{M} \to \mathfrak{M}_Z$, $\pi(w) = [w]$, is a surjective bounded morphism.*

Proof. It is obvious that π is surjective. Let us show that π is a bounded morphism.

(at_{bm}) We have: $w \in V(p)$ if and only if $[w] \in V_Z(p)$ if and only if $\pi(w) \in V_Z(p)$.
$(forth_{bm})$ Let $s \in \Sigma(w)$. Then $[s] \in \Sigma_Z([w])$, i.e. $\pi(s) \in \Sigma_Z(\pi(w))$ by the definition of bisimulation quotient.
$(back_{bm})$ Let $[s] \in \Sigma_Z(\pi(w))$, i.e. $[s] \in \Sigma_Z([w])$. By the definition of bisimulation quotient, there exist $w' \in [w]$ and s', $[s'] = [s]$, such that $s' \in \Sigma(w')$. By the $(back)$ condition, wZw' and $s' \in \Sigma(w')$ imply that there is t, $t\overline{Z}s'$, such that $t \in \Sigma(w)$. Now $[s'] = [s]$ and $[t] = [s']$ imply $[t] = [s]$. Thus, $\pi(t) = [s]$.

Hence, π is a surjective bounded morphism.

By the previous propositions, we obtain the following corollary.

Corollary 1. *Let $\mathfrak{M}_Z$ be the bisimulation quotient of an inquisitive modal model $\mathfrak{M}$. Then the states s and $[s]$ are modally equivalent. In particular, the worlds w and $[w]$ are modally equivalent.*

For the rest of this section we turn to the main result of the paper, also an analogue to a known result in modal logic (cf. [3]).

Definition 4. *We say that two inquisitive modal models $\mathfrak{M} = (W, \Sigma, V)$ and $\mathfrak{M}' = (W', \Sigma', V')$ are **globally bisimilar** if there exists a bisimulation X such that for all $w \in W$ there exists $w' \in W'$ such that wXw', and vice versa. In that case, we say that X is a **global bisimulation**.*

Definition 5. *Let $\mathfrak{M} = (W, \Sigma, V)$ and $\mathfrak{M}' = (W', \Sigma', V')$ be inquisitive modal models. A function $f : W \to W'$ is called an **isomorphism** of the models $\mathfrak{M}$ and $\mathfrak{M}'$ if f is a bijection and the following conditions are satisfied:*

(i) $w \in V(p)$ *if and only if* $f(w) \in V'(p)$, *for all* $p \in \mathrm{P}$;
(ii) $s \in \Sigma(w)$ *if and only if* $f(s) \in \Sigma'(f(w))$.

Lemma 1. *Let $\mathfrak{M} = (W, \Sigma, V)$ and $\mathfrak{M}' = (W', \Sigma', V')$ be inquisitive modal models and let Z and Z' be the largest bisimulations on $\mathfrak{M}$ and $\mathfrak{M}'$, respectively. Let $f : \mathfrak{M}_Z \to \mathfrak{M}'_{Z'}$ be a function defined with*

$$f([w]) = [w'] \text{ if there exist } x \in [w] \text{ and } x' \in [w'] \text{ such that } xXx',$$

where X is a global bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$. Then the function f is well-defined and, for given states s and s', we have

$$f([s]) = [s'] \text{ if and only if there exist } t,\ [t] = [s],\ \text{and } t',\ [t'] = [s'],\ \text{such that } t\overline{X}t'.$$

Proof. First we show that f is well-defined. Suppose $[u] = [v]$, i.e. uZv. We need to prove $f([u]) = f([v])$. Since X is a global bisimulation, there are some u' and v' such that uXu' and vXv'. Then we have $f([u]) = [u']$ and $f([v]) = [v']$, so it is sufficient to show $[u'] = [v']$, i.e. $u'Z'v'$. We have $u'X^{-1}uZvXv'$, so $u'(X^{-1} \circ Z \circ X)v'$. Thus, $u'Z'v'$ since Z' is the largest bisimulation on W'.

Now we show the second claim of the lemma. Let $f([s]) = [s']$. Suppose for a contradiction that for all $t, [t] = [s]$, and $t', [t'] = [s']$, it is not the case that $t\overline{X}t'$. In particular, for $t = s$ and $t' = s'$, it is not the case that $s\overline{X}s'$. That means that there exists $v \in s$ such that for all $v' \in s'$ it is not the case that vXv', i.e. $f([v]) \neq [v']$ or there exists $v' \in s'$ such that for all $v \in s$ it is not the case that vXv', i.e. $f([v]) \neq [v']$. Since $[v'] \in [s']$, for all $v' \in s'$ and $[v] \in [s]$, for all $v \in s$, we get a contradiction with $f([s]) = [s']$ in both cases.

Conversely, suppose that there are $t, [t] = [s]$, and $t', [t'] = [s']$, such that $t\overline{X}t'$. That means that for all $v \in t$ there exists $v' \in t'$ such that vXv', i.e. $f([v]) = [v']$, and vice versa.

Suppose for a contradiction that $f([s]) \not\subseteq [s']$ or $[s'] \not\subseteq f([s])$.

Let $f([s]) \not\subseteq [s']$. That means that there exists $[v'] \in f([s])$ such that $[v'] \notin [s']$. Then there is $[v] \in [s]$ such that $[v'] = f([v])$. Since $[v] \in [s] = [t]$, there is some $u \in t$ such that $[u] = [v]$, so we have $[v'] = f([u])$. Since $[v'] \notin [s'] = [t']$, it follows $f([u]) \notin [t']$, so for all $u' \in t'$ we get $[u'] \neq f([u])$, which contradicts $t\overline{X}t'$.

Let $[s'] \not\subseteq f([s])$. That means that there exists $[v'] \in [s']$ such that $[v'] \notin f([s])$, i.e. $[v'] \neq f([v])$, for all $[v] \in [s]$. Since $[v'] \in [s'] = [t']$, there is some $u' \in t'$ such that $[u'] = [v']$, so we have $[u'] \neq f([v])$, for all $[v] \in [s] = [t]$. Then for all $u \in t$ we get $[u'] \neq f([u])$, which contradicts $t\overline{X}t'$.

Theorem 1. *Let $\mathfrak{M} = (W, \Sigma, V)$ and $\mathfrak{M}' = (W', \Sigma', V')$ be inquisitive modal models and let Z and Z' be the largest bisimulations on $\mathfrak{M}$ and $\mathfrak{M}'$, respectively. Models $\mathfrak{M}$ and $\mathfrak{M}'$ are globally bisimilar if and only if the bisimulation quotients $\mathfrak{M}_Z$ and ${\mathfrak{M}'}_{Z'}$ are isomorphic.*

Proof. Let X be a global bisimulation between models $\mathfrak{M}$ and $\mathfrak{M}'$. Let $f : W_Z \to W'_{Z'}$ be the function defined with

$$f([w]) = [w'] \text{ if there exist } x \in [w] \text{ and } x' \in [w'] \text{ such that } xXx'.$$

By the previous lemma, the function f is well-defined.

We first show surjectivity of f. Let $[w'] \in W'_{Z'}$. Take an arbitrary $x' \in [w']$. Since X is a global bisimulation, there is some $w \in W$ such that wXx'. Since $w \in [w]$, it follows $f([w]) = [w']$.

Now, we show injectivity of f. Let $f([u]) = f([v])$. Then there are $x \in [u]$ and $x' \in f([v])$ such that xXx'. From $x \in [u]$ it follows uZx and, since $x' \in f([v])$, i.e. $[x'] = f([v])$, there exist $y \in [v]$ and $y' \in [x']$ such that yXy'. From $y \in [v]$ it follows yZv and, since $y' \in [x']$, we get $y'Z'x'$. Now we have $uZxXx'Z'^{-1}y'X^{-1}yZv$, so $u(Z \circ X \circ Z'^{-1} \circ X^{-1} \circ Z)v$. Thus uZv, since Z is the largest bisimulation on W. Hence, $[u] = [v]$.

It remains to prove (i) and (ii) to show that f is an isomorphism.

(i) Let $[w] \in V_Z(p)$. Then $w \in V(p)$. Since X is a global bisimulation, there is some $w' \in W'$ such that wXw'. Now we have $f([w]) = [w']$ and $w' \in V'(p)$, so $f([w]) \in V'_{Z'}(p)$. The converse is proved similarly.

(ii) Let $[s] \in \Sigma_Z([w])$. Then there are $x \in [w]$ and $t, [t] = [s]$, such that $t \in \Sigma(x)$. Since X is a global bisimulation, there is some $x' \in W'$ such that xXx', so by the $(forth)$ condition of the bisimulation X, there exists $t' \in \Sigma'(x')$ such that $t\overline{X}t'$.
Now $x \in [w]$ and xXx' imply $[x] = [w]$ and $f([x]) = [x']$, so we get $f([w]) = [x']$. Similarly, since $t\overline{X}t'$, by the previous lemma we get $f([t]) = [t']$, so from $[t] = [s]$ we have $f([s]) = [t']$.
Now from $x' \in f([w])$, $[t'] = f([s])$ and $t' \in \Sigma'(x')$ we get $f([s]) \in \Sigma'_{Z'}(f([w]))$.
Conversely, suppose $f([s]) \in \Sigma'_{Z'}(f([w]))$. Then there are $x' \in f([w])$ and $t', [t'] = f([s])$, such that $t' \in \Sigma'(x')$. Since X is a global bisimulation, there is some $x \in W$ such that xXx', so by the $(back)$ condition of the bisimulation X, there exists $t \in \Sigma(x)$ such that $t\overline{X}t'$.

Since $x' \in f([w])$, i.e. $[x'] = f([w])$, by the definition of f, there exist $y \in [w]$ and $y' \in [x']$ such that yXy'. Now we have $xXx'Z'y'X^{-1}yZw$, so $x(X \circ Z' \circ X^{-1} \circ Z)w$ and thus xZw since Z is the largest bisimulation on W.
Similarly, from $[t'] = f([s])$, by the previous lemma there exist $r, [r] = [s]$, and $r', [r'] = [t']$, such that $r\overline{X}r'$. Then we have $t\overline{X}t'\overline{Z'}r'\overline{X^{-1}}r\overline{Z}s$, so $t(\overline{X} \circ \overline{Z'} \circ \overline{X^{-1}} \circ \overline{Z})s$ and thus $t\overline{Z}s$ since Z is the largest bisimulation on W.
Now from xZw and $t\overline{Z}s$, it follows $x \in [w]$ and $[t] = [s]$ so, since $t \in \Sigma(x)$, we get $[s] \in \Sigma_Z([w])$.

For the converse, let $f : W_Z \to W'_{Z'}$ be an isomorphism of the models $\mathfrak{M}_Z$ and $\mathfrak{M}'_{Z'}$. We define a relation $X \subseteq W \times W'$ with

$$wXw' \text{ if and only if } f([w]) = [w'].$$

We claim that X is a global bisimulation.

Suppose $w \in W$. Then there is $w' \in W'$ such that $f([w]) = [w']$, so we get wXw'. Conversely, suppose $w' \in W'$. By the surjectivity of f, there exists $[w] \in W_Z$ such that $f([w]) = [w']$, so we get wXw'. This shows the globality of the relation X.

It remains to show that X is a bisimulation.

(at) Let wXw'. From definitions of bisimulation quotient and isomorphism, and from the claim $f([w]) = [w']$, it follows: $w \in V(p)$ if and only if $[w] \in V_Z(p)$ if and only if $f([w]) \in V'_{Z'}(p)$ if and only if $[w'] \in V'_{Z'}(p)$ if and only if $w' \in V'(p)$.

($forth$) Let wXw' and $s \in \Sigma(w)$. We need to find $s' \in \Sigma'(w')$ such that $s\overline{X}s'$.
By the definition of bisimulation quotient we have $[s] \in \Sigma_Z([w])$ and, since f is an isomorphism, it follows $f([s]) \in \Sigma'_{Z'}(f([w]))$, i.e. $f([s]) \in \Sigma'_{Z'}([w']))$.
That means that there exist $x \in [w']$ and $t \subseteq W'$, $[t] = f([s])$, such that $t \in \Sigma'(x)$. Since $x \in [w']$, we have $xZ'w'$, so, since $t \in \Sigma'(x)$, by the ($forth$) condition of the bisimulation Z', there exists $s' \in \Sigma'(w')$ such that $t\overline{Z'}s'$.
Now, $t\overline{Z'}s'$ implies $[t] = [s']$, so by $[t] = f([s])$, we get $[s'] = f([s])$.
It remains to show $s\overline{X}s'$. Suppose $v \in s$. Then $[v] \in [s]$, so we have $f([v]) \in f([s])$. Now we have $f([v]) \in [s']$, so there exists $[u'] \in [s']$ such that $f([v]) = [u']$. Since $[u'] \in [s']$, there exists $v' \in s'$ such that $[u'] = [v']$, so we get $f([v]) = [v']$. Hence, vXv'. The converse can be proved similarly, so we have $s\overline{X}s'$.

($back$) Similarly as the condition ($forth$).

5 Future work

There are several directions for further work. First, we could examine if inquisitive modal logic has a finite model property, i.e. if a formula is satisfiable on some model, then it is satisfiable on a finite model. One of the possible ways to see this could be via filtration of a model. Second, it would be interesting to characterize frame definability of InqML, i.e. to provide an analogue of the Goldblatt-Thomason theorem for inquisitive modal logic (cf. [4]).

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