Rule-Elimination Theorems

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Abstract. In this paper, we give a non-algorithmic proof of the CUT-elimination theorem for the propositional fragment of **LK**. We then investigate the essential properties that make the proof work and find that it is not something particular to the rule CUT but dependent on the properties of the sequent system under consideration. We next introduce the notion of *normal sequent structures* and show that we can eliminate certain rules in this framework. Analyzing the proof, we again point out the essential properties that make the proof work, and based on these observations formulate the notion of *abstract sequent structures*. For these class of structures, we prove our main theorems: the necessary and sufficient conditions for eliminating any structural rule.

Keywords: sequent salculus · CUT-elimination theorem · universal logic · rule-elimination theorem

1 Introduction

CUT-elimination theorems constitute one of the most important class of theorems of proof theory and have many important consequences. Since Gentzen's proof of the CUT-elimination theorem for the system **LK** − a calculus for the First-order Classical Logic − introduced in [4], several other proofs of the theorem has been proposed (see [5]). Even though the techniques of these proofs can be modified to sequent systems other than **LK**, they are essentially of a very particular nature; each of them describes an algorithm to transform a given proof to a CUT-free proof.

CUT-elimination can, however, be seen as the *elimination* of a certain rule. One may, therefore, ask the same question for any rule in any sequent system. We can begin this investigation with the following questions.

(1) What makes the elimination of CUT possible in **LK**? Do the other rules play any part?
(2) Is it possible to characterize sequent systems for which CUT-elimination holds?

* I would like to thank my advisors Prof. Sankha S. Basu and Prof. Mihir K. Chakraborty for encouragement and suggesting modifications of this article. I would also like to thank Department of Mathematics, Indraprastha Institute of Information Technology, Delhi for generously supporting the project and the trip facilitating the presentation of the same at ESSLLI.

(3) Is it possible to give necessary and sufficient conditions of eliminating *any* rule from a given sequent system? What does a 'rule' mean? What are we supposed to understand by a 'sequent system'?

Unfortunately, the algorithmic proofs of the CUT-elimination theorems hardly shed any light on issues like the above, primarily due to their heavy dependence on the syntactic structures of the rules.

We, therefore, consider rules abstractly, within the framework of logical structures familiar from *universal logic* in the sense of [1]. A logical structure is a pair of the form $(\mathscr{L}, \vdash)$ where $\mathscr{L}$ is a set and $\vdash \subseteq \mathcal{P}(\mathscr{L}) \times \mathscr{L}$. In particular, $\mathscr{L}$ can be a set of 'sequents', and $\vdash$ can be defined so that, given any sequent system $\mathbf{S}$, $\emptyset \vdash_{\mathbf{S}} \Gamma \implies \Delta$ holds whenever there is a proof of the sequent $\Gamma \implies \Delta$ in $\mathbf{S}$. We can thus connect the theory of logical structures from universal logic with proof-theory.

One of the goals of universal logic, according to [2], is "to go beyond particular logical systems to clarify the fundamental concepts of logic and to construct general proofs." In this paper, our aim is to clarify the essence of the so-called "ELIMINATION THEOREMS" and construct general proofs of the same. We will do the former by providing a proof of the CUT-elimination theorem for **PLK**, the propositional fragment of **LK**, and the latter by proving what we call RULE-elimination theorems.

In §1, we give a non-algorithmic proof of the CUT-elimination theorem for **PLK**. From this proof, we abstract the essential features of the argument and define *normal sequent structures relative to a particular rule*. We then prove a version of the RULE-elimination theorem for these in §2. In §3, we define the notion of *abstract sequent structures* and point out the essential features that made the proof of the RULE-elimination theorem for the normal sequent structures work. This paves the way towards formulating the most general version of the RULE-elimination theorems. We then show that for abstract sequent structures, the RULE-elimination theorem also has a converse: thus answering question (3) above. Finally, in the concluding section, we point to some directions for further research.

2 Cut-elimination theorem for the Propositional Fragment of LK

In this section, we provide a non-algorithmic proof of the CUT-elimination theorem for **PLK**. This is motivated by Buss's proof of CUT-elimination theorem for a variant of **LK** in [3]. In what follows, by CUT-rule, we mean the following.

$$\frac{\Gamma \implies \Delta, A \qquad A, \Gamma \implies \Delta}{\Gamma \implies \Delta}\ \text{CUT}$$

By the CONTRACTION-rules, we mean both the following rules.

$$\frac{A, A, \Gamma \implies \Delta}{A, \Gamma \implies \Delta}\ \mathsf{CL} \qquad\qquad \frac{\Gamma \implies \Delta, A, A}{\Gamma \implies \Delta, A}\ \mathsf{CR}$$

Here CL and CR stand for the left and the right contraction rules, respectively.

Lemma 1. *Let $\Gamma \implies \Delta$ be the end sequent of a* **PLK***-proof. Suppose also that it has been obtained by an application of a rule, R. Then each premise of R also has a* **PLK***-proof.*

Lemma 2. *Let $\Gamma \implies \Delta$ be the end sequent of a* **PLK***-proof. Suppose also that it has been obtained by an application of a rule, R. If this proof of $\Gamma \implies \Delta$ is* CONTRACTION+CUT*-free, then each premise of R also has a* CONTRACTION+CUT*-free* **PLK***-proof.*

Lemma 3. *Let $\Gamma \implies \Delta$ be the end sequent of a* **PLK***-proof. Then at least one of Γ, Δ is not the empty list.*

For the following result we will need to interpret the language by means of *truth assignments*, these being mappings of the form $v : P_{\mathbf{PLK}} \to \{0, 1\}$ where $P_{\mathbf{PLK}}$ denotes the set of propositional variables of the particular language of **PLK** under consideration (generally taken to be infinite). We then extend this mapping to the set of all formulas, denoting it by $\overline{v} : \mathscr{L}_{\mathbf{PLK}} \to \{0, 1\}$, recursively as follows:

$$\begin{aligned} \overline{v}(p) &= v(p) & \text{for all } p \in P_{\mathbf{PLK}} \\ \overline{v}(\neg\alpha) &= 1 - \overline{v}(\alpha) \\ \overline{v}(\alpha \wedge \beta) &= \min\{\overline{v}(\alpha), \overline{v}(\beta)\} \\ \overline{v}(\alpha \vee \beta) &= \max\{\overline{v}(\alpha), \overline{v}(\beta)\} \end{aligned}$$

and calling such a mapping a *valuation*. Given a valuation $\overline{v} : \mathscr{L}_{\mathbf{PLK}} \to \{0, 1\}$ where and a **PLK**-sequent $\Gamma \implies \Delta$, we say that $\overline{v}$ *satisfies* $\Gamma \implies \Delta$ if there exists $\varphi \in \Gamma$ such that $\overline{v}(\varphi) = 0$ or there exists $\psi \in \Delta$ such that $\overline{v}(\psi) = 1$. A **PLK**-sequent $\Gamma \implies \Delta$ is said to be *valid* if every valuation satisfies $\Gamma \implies \Delta$.

Theorem 1 (Soundness Theorem for PLK). *Every **PLK**-provable sequent is **PLK**-valid.*

The proof of Lemma 3 is now easy. Suppose that $\Gamma \implies \Delta$ be the end sequent of a **PLK**-proof but that both Γ as well as Δ are empty lists. Note that since $\Gamma \implies \Delta$ is **PLK**-provable, it is **PLK**-valid as well. Now, choose any $p \in P_{\mathbf{PLK}}$ and define a truth assignment $w : P_{\mathbf{PLK}} \to \{0, 1\}$ as follows:

$$w(q) = \begin{cases} 1 & \text{if } q = p \\ 0 & \text{else} \end{cases}$$

Let $\overline{w}$ be the corresponding valuation. Note that since $\Gamma \implies \Delta$ is **PLK**-valid, it follows that either there exists $\varphi \in \Gamma$ such that $\overline{v}(\varphi) = 0$ or there exists $\psi \in \Delta$ such that $\overline{v}(\psi) = 1$. But both of them are impossible since both Γ and Δ are empty. This contradiction ensures our conclusion.

Theorem 2 (Cut-Elimination Theorem for PLK (Part I)). *Let $\Gamma \Longrightarrow \Delta$ be a* **PLK**-*provable sequent. Then there exists a* CONTRACTION-*free* **PLK**-*proof of $\Gamma \Longrightarrow \Delta$.*

Theorem 3 (Cut-Elimination Theorem for PLK (Part II)). *Let $A \to A, \Gamma \Longrightarrow \Delta$ be a* ***PLK***-*provable sequent. If $A \to A, \Gamma \Longrightarrow \Delta$ has a* CONTRACTION+CUT-*free* ***PLK***-*proof, then so does $\Gamma \Longrightarrow \Delta$.*

We now give outlines of the proofs of the above theorems. The proof of Theorem 2 is routine. The proof of Theorem 3 uses induction on the complexity of the sequent $\Gamma \Longrightarrow \Delta$ (i.e., the total number of connectives in $\Gamma \Longrightarrow \Delta$). The crucial property here is that the complexity of the conclusion sequent of a logical rule is always strictly greater than that(those) of the premise sequent(s). To make the induction step work, we consider the conclusion sequent and rise up from the bottom to stop at the first sequent that is the conclusion of a logical rule. Then, depending on the occurrence of $A \to A$ before or in this sequent, and considering each logical rule, we prove the theorems.

We now make the following observations.

(1) The proof of CUT-elimination outlined above cannot be applied as it is to prove CUT-elimination for a system in which either the WEAKENING- and/or the EXCHANGE-rules are restricted. This, however, does not rule out the possibility of eliminating some restricted form of CUT by applying the same technique.
(2) As mentioned earlier, in each case of the proof of Theorem 2, the only thing that made the induction work, was the fact that the complexity of each of the premise sequents of a logical rule is strictly less than that of the conclusion sequent. It may be noted that the number of premises has no role.
(3) We note that the notion of sequent complexity is independent of connectives. Hence the proof can be adapted to other sequent systems. However, as we have pointed out later, we can generalize the argument further.
(4) The proof of Theorem 2 uses the crucial property that the sequent $\emptyset \Longrightarrow \emptyset$ is not valid. It also uses the fact that if a sequent $\Gamma \Longrightarrow \Delta$ is valid and contains only propositional variables, then there is a propositional varaible p common to both of them.

3 Rule-elimination Theorem for Normal Sequent Structures

In this section, we generalize the above argument to a broader class of sequent systems, taking note of the above observations. We introduce the notion of *normal sequent structures* and prove a "RULE-elimination theorem" for the same. But before that, we define precisely what we mean by *a rule* and what it means to *eliminate a rule.*

Definition 1. *Let $\mathscr{L}$ be a set and $\Longrightarrow \subseteq \boldsymbol{FO}(\mathscr{L}) \times \boldsymbol{FO}(\mathscr{L})$, where $\boldsymbol{FO}(\mathscr{L})$ is the set of all finite ordered lists of elements in $\mathscr{L}$. Then $(\Gamma, \Delta) \in \Longrightarrow$ is said* *to be an* $\mathscr{L}$-sequent with respect to $\Longrightarrow$. *We will write* $\Gamma \Longrightarrow \Delta$, *instead of* $(\Gamma, \Delta) \in \Longrightarrow$. *The set of all* $\mathscr{L}$*-sequents with respect to* $\Longrightarrow$, *will be denoted by* $\mathsf{Seq}(\mathscr{L}, \Longrightarrow)$.[1]

Definition 2. *Let* $\mathsf{Seq}(\mathscr{L}, \Longrightarrow)$ *be the set of all* $\mathscr{L}$*-sequents with respect to* $\Longrightarrow$. *A set* R *is said to be a* $\mathsf{Seq}(\mathscr{L}, \Longrightarrow)$-non-axiomatic rule of inference *if*

$$\emptyset \subsetneq R \subseteq \mathsf{Seq}(\mathscr{L}, \Longrightarrow)^n \times \mathsf{Seq}(\mathscr{L}, \Longrightarrow)$$

for some $n \in \mathbb{N}$. *It is said to be a* $\mathsf{Seq}(\mathscr{L}, \Longrightarrow)$-axiomatic rule of inference *if* $\emptyset \subset R \subseteq \{\emptyset\} \times \mathsf{Seq}(\mathscr{L}, \Longrightarrow)$. R *is said to be a* $\mathsf{Seq}(\mathscr{L}, \Longrightarrow)$-rule of inference *if it is either a* $\mathsf{Seq}(\mathscr{L}, \Longrightarrow)$*-non-axiomatic rule of inference or an* $\mathsf{Seq}(\mathscr{L}, \Longrightarrow)$*-axiomatic rule of inference.*

Definition 3 (Pre-Normal Sequent Structure). *Let* $\mathscr{L}$ *be the formula algebra over some language,* $\mathsf{Seq}(\mathscr{L}, \Longrightarrow)$ *be set of all* $\mathscr{L}$*-sequents with respect to* $\Longrightarrow$ *, and* $\vdash \subseteq \mathcal{P}(\mathsf{Seq}(\mathscr{L}, \Longrightarrow)) \times \mathsf{Seq}(\mathscr{L}, \Longrightarrow)$. *Let* $\mathscr{P}$ *be the set of propositional variables. The pair* $(\mathsf{Seq}(\mathscr{L}, \Longrightarrow), \vdash)$ *is then said to be a* pre-normal sequent structure *if the following conditions hold.*

(1) $\alpha \Longrightarrow \alpha$ *for all* $\alpha \in \mathscr{P}$.
(2) Both the WEAKENING *and* EXCHANGE *rules are given, i.e., we have,*

$$\frac{\Gamma \Longrightarrow \Delta}{A, \Gamma \Longrightarrow \Delta}\ \mathsf{WL} \qquad\qquad \frac{\Gamma \Longrightarrow \Delta}{\Gamma \Longrightarrow \Delta, A}\ \mathsf{WR}$$

WL *and* WR *denoting respectively the left and right weakening rule; and,*

$$\frac{\Pi, A, B, \Gamma \Longrightarrow \Delta}{\Pi, B, A, \Gamma \Longrightarrow \Delta}\ \mathsf{EL} \qquad\qquad \frac{\Gamma \Longrightarrow \Delta, A, B, \Lambda}{\Gamma \Longrightarrow \Delta, B, A, \Lambda}\ \mathsf{ER}$$

EL *and* ER *denoting respectively the left and right exchange rule.*
(3) $\emptyset \vdash \Gamma \Longrightarrow \Delta$ *means that there is a proof of* $\Gamma \Longrightarrow \Delta$ *in the 'usual' sense. (See, e.g., the one given in §§1.3 of [5].) In this case we say that* $\Gamma \Longrightarrow \Delta$ *is* $\mathsf{Seq}(\mathscr{L}, \Longrightarrow)$*-provable or provable in* $\mathsf{Seq}(\mathscr{L}, \Longrightarrow)$.
(4) If $\Gamma \Longrightarrow \Delta$ *is provable in* $\mathsf{Seq}(\mathscr{L}, \Longrightarrow)$ *and if* Γ, Δ *both contain only propositional variables, then there exists* $p \in \mathscr{P}$ *such that* $p \in \Gamma \cap \Delta$.
(5) If $\Gamma \Longrightarrow \Delta$ *is provable in* $\mathsf{Seq}(\mathscr{L}, \Longrightarrow)$ *then either* Γ *is not the empty list or* Δ *is not the empty list.*

Definition 4 (Witness Function). *Let* $(\mathsf{Seq}(\mathscr{L}, \Longrightarrow), \vdash)$ *be a pre-normal sequent structure and* $v : \mathsf{Seq}(\mathscr{L}, \Longrightarrow) \to \mathbb{N}$. *Let* T *be a* $\mathsf{Seq}(\mathscr{L}, \Longrightarrow)$*-non-axiomatic rule. Then* v *is said to be a* witness function for $(\mathsf{Seq}(\mathscr{L}, \Longrightarrow), \vdash)$ relative to the rule T *if the following conditions hold.*

[1] It may be noted that we could have used $\Longrightarrow$ as the set of all sequents. However, the traditional way of representing sequents does not support that. Hence we have used an additional set, viz., $\mathsf{Seq}(\mathscr{L}, \Longrightarrow)$.

(1) *There exists* $n_0 \in \mathbb{N}$ *such that if for some sequent* $\Gamma \Longrightarrow \Delta$, $v(\Gamma \Longrightarrow \Delta) = n_0$ *then* Γ, Δ *contain only propositional variables. Moreover,* $v(\Gamma \Longrightarrow \Delta) \geq n_0$, *for all* $\Gamma \Longrightarrow \Delta \in \mathsf{Seq}(\mathscr{L}, \Longrightarrow)$.

(2) *Suppose* R *is a* $\mathsf{Seq}(\mathscr{L}, \Longrightarrow)$*-non-axiomatic rule. We say that* v respects (strictly respects) *an instance* $((\Gamma_1 \Longrightarrow \Delta_1, \ldots, \Gamma_m \Longrightarrow \Delta_m), \Gamma \Longrightarrow \Delta)$ *of* R *if* $v(\Gamma_i \Longrightarrow \Delta_i) \leq v(\Gamma \Longrightarrow \Delta)$ *(resp., if* $v(\Gamma_i \Longrightarrow \Delta_i) < v(\Gamma \Longrightarrow \Delta)$*), for all* $i \in \{1, \ldots, m\}$*. If this holds for every instance of a rule* R*, then* R *is called a* $\mathsf{Seq}(\mathscr{L}, \Longrightarrow)$-logical rule. v *does not respect any instance of* T.

Definition 5. *Let* $(\mathsf{Seq}(\mathscr{L}, \Longrightarrow), \vdash)$ *be a pre-normal sequent structure. It is said to be a* normal sequent structure relative to a rule T *if there exists a witness function* v *for* $(\mathsf{Seq}(\mathscr{L}, \Longrightarrow), \vdash)$ *relative to* T*, and for every* $\mathsf{Seq}(\mathscr{L}, \Longrightarrow)$*-provable sequent* $\Gamma \Longrightarrow \Delta$*, followings hold.*

(1) *If* Γ *is not the empty list, then there exist a rule* R *and a sequent* $\Gamma^R \Longrightarrow \Delta^R$ *such that* $\Gamma \Longrightarrow \Delta$ *is one of the premise(s) and* $\Gamma^R \Longrightarrow \Delta^R$ *the conclusion, of some instance of* R*. If* Δ *is not the empty list, then there exist a rule* L *and a sequent* $\Gamma^L \Longrightarrow \Delta^L$ *such that* $\Gamma \Longrightarrow \Delta$ *is one of the premise(s) and* $\Gamma^L \Longrightarrow \Delta^L$ *the conclusion, of some instance of* R.

(2) $v(\Gamma' \Longrightarrow \Delta') < v(\Gamma^R \Longrightarrow \Delta^R)$ *for every premise of* R *and* $v(\Gamma'' \Longrightarrow \Delta'') < v(\Gamma^L \Longrightarrow \Delta^L)$ *for every premise of* L.

(3) *If there exists a proof of* $\Gamma^R \Longrightarrow \Delta^R$*, then there exists a* T*-free* $\mathsf{Seq}(\mathscr{L}, \Longrightarrow)$*-proof of* $\Gamma \Longrightarrow \Delta$*. Similarly, if there exists a proof of* $\Gamma^L \Longrightarrow \Delta^L$*, then there exists a* T*-free* $\mathsf{Seq}(\mathscr{L}, \Longrightarrow)$*-proof of* $\Gamma \Longrightarrow \Delta$.

Theorem 4. *Let* $(\mathsf{Seq}(\mathscr{L}, \Longrightarrow), \vdash)$ *be a normal sequent structure relative to a* $\mathsf{Seq}(\mathscr{L}, \Longrightarrow)$*-rule* T *that is disjoint from* WEAKENING *and* EXCHANGE*. Suppose* $\Gamma \Longrightarrow \Delta$ *is* $\mathsf{Seq}(\mathscr{L}, \Longrightarrow)$*-provable. Then there exists a* T*-free* $\mathsf{Seq}(\mathscr{L}, \Longrightarrow)$*-proof of* $\Gamma \Longrightarrow \Delta$.

Proof. Note that since $\Gamma \Longrightarrow \Delta$ is provable then by Definition 3(5) it follows that either Γ is not the empty list or Δ is not the empty list. We first consider the case when Γ is not the empty list. Since $(\mathsf{Seq}(\mathscr{L}, \Longrightarrow), \vdash)$ is a normal sequent structure, there exists a logical rule R and a sequent $\Gamma^R \Longrightarrow \Delta^R$ such that $\Gamma \Longrightarrow \Delta$ is one of the premise(s), and $\Gamma^R \Longrightarrow \Delta^R$ the conclusion, of an instance of R. Furthermore, $v(\Gamma' \Longrightarrow \Delta') < v(\Gamma^R \Longrightarrow \Delta^R)$ for every premise of R. This implies that $\Gamma^R \Longrightarrow \Delta^R$ has a proof and consequently by Definition 5(3), there exists a T-free $\mathsf{Seq}(\mathscr{L}, \Longrightarrow)$-proof of $\Gamma \Longrightarrow \Delta$. Then case when Δ is not the empty list can be dealt with similarly.

Remark 1. The rule T could be CUT, or some restricted version of it. In general, T could be any rule as long as it is disjoint from WEAKENING and EXCHANGE.

4 Rule-elimination Theorem for Abstract Sequent Structures

In the previous section, we discussed the elimination of a rule T in a normal sequent structure relative to T. We can, however, generalize further by dropping the requirement that the sequents be constructed based on some algebraic language.

Since the elements of the sequents are no longer members of an algebra, the rules that can potentially be eliminated can be purely "structural" in nature. It is, however, trivial to show, e.g., that if there is a logical rule that introduces a new connective for which there is no logical rule that removes it, then such a rule cannot be eliminated (assuming the usual definition of proofs).

We start with a more general definition of sequent structures. To motivate this, we analyze the proof of Theorem 4 from the previous section as follows.

(1) Even though the witness function uses $\mathbb{N}$, no particular property of it was used except for the fact that it is a poset.
(2) We have assumed the usual notion of proof as is standard in the literature on sequent calculus. However, the actual requirements were limited to results such as the Lemmas 1, 2, and 3.
(3) The existence of rules R and L in Definition 5 were crucial in proving Theorem 4.

Definition 6. *An* abstract sequent structure *is a triple* $\mathbf{AbSS} = \big(\mathsf{Seq}(\mathscr{L}, \Longrightarrow), \vdash\big)$*, where* $\mathsf{Seq}(\mathscr{L}, \Longrightarrow)$ *denotes the set of all* $\mathscr{L}$*-sequents with respect to* $\Longrightarrow$ *for some set* $\mathscr{L}$ *and* $\vdash \subseteq \mathcal{P}(\mathsf{Seq}(\mathscr{L}, \Longrightarrow)) \times \mathsf{Seq}(\mathscr{L}, \Longrightarrow)$*.*

Given an abstract sequent structure, **AbSS**, an **AbSS**-*rule of inference* is defined exactly in the same way as in Definition 2.

Definition 7. *Given an abstract sequent structure* $\mathbf{AbSS} = \big(\mathsf{Seq}(\mathscr{L}, \Longrightarrow), \vdash\big)$ *and a family* $\mathcal{R} = \{R_i\}_{i\in I}$ *of* **AbSS***-rules of inference, the* abstract sequent structure induced by $\{R_i\}_{i\in I}$ *is a triple,* $\mathbf{AbSS}_{\mathcal{R}} = \big(\mathsf{Seq}(\mathscr{L}, \Longrightarrow), \vdash_{\mathcal{R}}\big)$ *where* $\vdash_{\mathcal{R}}$ *is such that the following conditions hold.*

(1) $\emptyset \vdash_{\mathcal{R}} \Gamma \Longrightarrow \Delta$ *if* $(\emptyset, \Gamma \Longrightarrow \Delta) \in R_i$ *whenever* R_i *is an* **AbSS***-axiomatic rule.*
(2) $\emptyset \vdash_{\mathcal{R}} \Gamma \Longrightarrow \Delta$ *implies that there exists* R_i *such that* $(\emptyset, \Gamma \Longrightarrow \Delta) \in R_i$ *where* R_i *is a* **AbSS***-axiomatic rule, or* $((\Gamma_j \Longrightarrow \Delta_j)_{j\in J}, \Gamma \Longrightarrow \Delta) \in R_i$ *for some family of sequents* $(\Gamma_j \Longrightarrow \Delta_j)_{j\in J}$*.*
(3) For any **AbSS***-rule* R_i *which is not* **AbSS***-axiomatic the following statement holds. If* $((\Gamma_j \Longrightarrow \Delta_j)_{j\in J}, \Gamma \Longrightarrow \Delta) \in R_i$ *then* $\emptyset \vdash_{\mathcal{R}} \Gamma \Longrightarrow \Delta$ *iff* $\emptyset \vdash_{\mathcal{R}} \Gamma_j \Longrightarrow \Delta_j$ *for all* $j \in J$*.*

Remark 2. The above properties assert the following: (1) axiomatic inferences are permitted in an **AbSS**; (2) if a sequent is provable, then it is either the conclusion of an axiomatic rule or is obtained by using some non-axiomatic rule; and (3) if the conclusion of some rule has a proof, then so does each of its premises (cf. Lemma 1).

Given an abstract sequent structure $\mathbf{AbSS} := \big(\mathsf{Seq}(\mathscr{L}, \Longrightarrow), \vdash\big)$ and a family $\mathcal{R} := \{R_i\}_{i\in I}$ of **AbSS**-rules, **AbSS** is said to be *generated* by $\mathcal{R}$ if $\vdash = \vdash_{\mathcal{R}}$.

Definition 8. *Given an abstract sequent structure* $\mathbf{AbSS} = \big(\mathsf{Seq}(\mathscr{L}, \Longrightarrow), \vdash\big)$ *generated by a family* $\mathcal{R} = \{R_i\}_{i \in I}$ *of* **AbSS***-rules, an* **AbSS***-rule of inference* R_i *is said to be* **AbSS-**eliminable *if* $\emptyset \vdash \Gamma \Longrightarrow \Delta$ *implies that* $\emptyset \vdash_{\mathcal{R}\setminus\{R_i\}} \Gamma \Longrightarrow \Delta$.

Definition 9. *Let* $\mathbf{AbSS} = \big(\mathsf{Seq}(\mathscr{L}, \Longrightarrow), \vdash\big)$ *be an abstract sequent structure generated by a family* $\mathcal{R} = \{R_i\}_{i \in I}$ *of* **AbSS***-rule(s) of inference. Let* $(P, \leq)$ *be a poset and* $v : \mathsf{Seq}(\mathscr{L}, \Longrightarrow) \to P$*. Then* v *is said to* respect (strictly respect) *an instance* $((\Gamma_j \Longrightarrow \Delta_j)_{j \in J}, \Gamma \Longrightarrow \Delta) \in R_i$ *of the* **AbSS***-rule* R_i*, if* $v(\Gamma_j \Longrightarrow \Delta_j) \leq v(\Gamma \Longrightarrow \Delta)$ *(resp., if* $v(\Gamma_j \Longrightarrow \Delta_j) < v(\Gamma \Longrightarrow \Delta)$*) for all* $j \in J$.

Theorem 5 (Rule-Elimination Theorem). *Let* $\mathbf{AbSS} = \big(\mathsf{Seq}(\mathscr{L}, \Longrightarrow), \vdash_{\mathcal{R}}\big)$ *be an abstract sequent structure generated by a family* $\mathcal{R} = \{R_i\}_{i \in I}$ *of* **AbSS***-rules,* $(P, \leq)$ *be a poset, and* $v : \mathsf{Seq}(\mathscr{L}, \Longrightarrow) \to P$*. Suppose* R_k *is an* **AbSS***-non-axiomatic rule such that the following conditions hold.*

(1) $R_i \cap R_k = \emptyset$ *for all* $R_i \in \mathcal{R} \setminus \{R_k\}$
(2) For every $((\Gamma_j \Longrightarrow \Delta_j)_{j \in J}, \Gamma \Longrightarrow \Delta) \in R_k$*,* $v(\Gamma_j \Longrightarrow \Delta_j) > v(\Gamma \Longrightarrow \Delta)$ *for all* $j \in J$*.*
(3) For every **AbSS***-axiomatic rule* R_i *and for all* $(\emptyset, \Gamma \Longrightarrow \Delta) \in R_i$*, if* $\emptyset \vdash_{\mathcal{R}} \Gamma \Longrightarrow \Delta$ *then* $\emptyset \vdash_{\mathcal{R}\setminus\{R_k\}} \Gamma \Longrightarrow \Delta$*.*
(4) If $\emptyset \vdash_{\mathcal{R}} \Gamma \Longrightarrow \Delta$ *then either* $(\emptyset, \Gamma \Longrightarrow \Delta) \in R_i$ *for some* **AbSS***-axiomatic rule or* $v(\Gamma_j \Longrightarrow \Delta_j) \not> v(\Gamma \Longrightarrow \Delta)$ *for every* $j \in J$ *with* $((\Gamma_j \Longrightarrow \Delta_j)_{j \in J}, \Gamma \Longrightarrow \Delta) \in R_i$ *for some* **AbSS***-rule* R_i*.*
(5) If $v(\Gamma_j \Longrightarrow \Delta_j) \not> v(\Gamma \Longrightarrow \Delta)$ *for every* $j \in J$ *with* $((\Gamma_j \Longrightarrow \Delta_j)_{j \in J}, \Gamma \Longrightarrow \Delta) \in R_i$ *for some* **AbSS***-rule* R_i*, and* $\emptyset \vdash_{\mathcal{R}} \Gamma_j \Longrightarrow \Delta_j$ *for all* $j \in J$*, then* $\emptyset \vdash_{\mathcal{R}\setminus\{R_k\}} \Gamma_j \Longrightarrow \Delta_j$ *for all* $j \in J$*.*

Then R_k *is* **AbSS***-eliminable.*

Theorem 6. *Let* $\mathbf{AbSS} = \big(\mathsf{Seq}(\mathscr{L}, \Longrightarrow), \vdash_{\mathcal{R}}\big)$ *be an abstract sequent structure generated by a family* $\mathcal{R} := \{R_i\}_{i \in I}$ *of* **AbSS***-rules. Let* R_k *be an* **AbSS***-non-axiomatic rule which is* **AbSS***-eliminable and is disjoint from every rule in* $\mathcal{R} \setminus \{R_k\}$*. Then there exists a function* $v : \mathsf{Seq}(\mathscr{L}, \Longrightarrow) \to P$ *for some poset* $(P, \leq)$ *such that the following conditions hold.*

(1) For every $((\Gamma_j \Longrightarrow \Delta_j)_{j \in J}, \Gamma \Longrightarrow \Delta) \in R_k$*,* $v(\Gamma_j \Longrightarrow \Delta_j) > v(\Gamma \Longrightarrow \Delta)$ *for all* $j \in J$*.*
(2) For every **AbSS***-axiomatic rule* R_i *and for all* $(\emptyset, \Gamma \Longrightarrow \Delta) \in R_i$*, if* $\emptyset \vdash_{\mathcal{R}} \Gamma \Longrightarrow \Delta$ *then* $\emptyset \vdash_{\mathcal{R}\setminus\{R_k\}} \Gamma \Longrightarrow \Delta$*.*
(3) If $\emptyset \vdash_{\mathcal{R}} \Gamma \Longrightarrow \Delta$ *then either* $(\emptyset, \Gamma \Longrightarrow \Delta) \in R_i$ *for some* **AbSS***-axiomatic rule or* $v(\Gamma_j \Longrightarrow \Delta_j) \not> v(\Gamma \Longrightarrow \Delta)$ *for every* $j \in J$ *with* $((\Gamma_j \Longrightarrow \Delta_j)_{j \in J}, \Gamma \Longrightarrow \Delta) \in R_i$ *for some* **AbSS***-rule* R_i*.*
(4) If $v(\Gamma_j \Longrightarrow \Delta_j) \not> v(\Gamma \Longrightarrow \Delta)$ *for every* $j \in J$ *with* $((\Gamma_j \Longrightarrow \Delta_j)_{j \in J}, \Gamma \Longrightarrow \Delta) \in R_i$ *for some* **AbSS***-rule* R_i *and* $\emptyset \vdash_{\mathcal{R}} \Gamma_j \Longrightarrow \Delta_j$ *for all* $j \in J$ *then* $\emptyset \vdash_{\mathcal{R}\setminus\{R_k\}} \Gamma_j \Longrightarrow \Delta_j$ *for all* $j \in J$*.*

5 Concluding Remarks

In this paper a necessary and sufficient condition of eliminating structural rules from any sequent system (to be precise, from *abstract sequent structures*) has been provided. While this is interesting from the theoretical perspective, questions can be raised about the applicability of such an abstract result. It would be great if we could find a necessary and sufficient condition that is, so to speak, 'immediately applicable'. But that is left for the future. A part of the reason why CUT-elimination theorem is so important has to do with it consequences, such as the subformula property, decidability etc. that such an elimination entails. No such result could have been proved from the RULE-elimination theorem. We would like to work on this in future.

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