# Forcing for an Optimal $A$-translation

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**Abstract.** Kripke semantics for intuitionistic predicate logic IQC is often viewed as a forcing relation between posets and formulas [10, 6, 2]. In this paper, we further introduce Cohen forcing into semantics [11, 3]. In particular, we use generic filters to interpret the double-negation translations from classical first-order logic to the intuitionistic version. It explains how our method interprets classical theories into constructive ones [14, 1]. In addition, our approach is generalized to Friedman's $A$-translation [7]. Consequently, we propose an optimal $A$-translation that extends the class of theorems that are conserved from a classical theory to it intuitionistic counterpart.

**Keywords:** Intuitionistic logic · double-negation translation · Friedman's trick · Cohen forcing.

## 1 Introduction

Intuitionistic first-order logic IQC includes every axiom and inference rule of classical first-order logic CQC except the law of excluded middle. Thus the latter can be regarded as an extension of the former, and the two theories are equiconsistent. However, they are not compatible; there are formulas consistent with IQC while contradicting CQC, for instance: $\neg\forall x(Ax \vee \neg Ax)$ [14, 13].

Attempts to reconcile intuitionistic and classical theories are fruitful. In particular, one can interpret a classical theory into its intuitionistic counterparts using elaborate translations, such as double-negation translations from Peano arithmetic PA to Heyting arithmetic HA. The translation not only translates the theoremhood, but also transforms classical proofs into intuitionistic ones. In addition, we can optimize the translation if we restrict the formulas to fragments of arithmetic [1, 4]; for instance, the $\Pi_2$-fragment of PA is conserved in HA [7, 12]. In this paper, we study variants of double-negation translations, and semantically generalize them to Friedman's $A$-translation. The method is adapted from Cohen forcing, one he uses to construct models satisfying the negation of the continuum hypothesis and of the axiom of choice [3]. From an optimal $A$-translation, we obtain a large class of formulas – preservable formula – which provability is conserved between HA and PA.

We briefly outline the paper. In Section 2 we fix the notation for the Kripke semantics of IQC and CQC. Section 3 introduces the necessary semantical tool

– generic set – that is used in Section 4 to interpret variants of $A$-translation. Section 5 formulates a translation that inserts much fewer double $A$-implications than the ones known in the literature, and following from that in Section 6 we obtain a class of formulas that are conserved after the optimal $A$-translation, that is larger than the usual $\Pi_2$-fragment.

## 2 Forcing

Formulas of IQC and CQC are defined as usual, and they are interpreted on the worlds of *Kripke models*, which is a tuple $(\mathbb{P}, \leqslant, V, R, D)$, where $(\mathbb{P}, \leqslant)$ is a *partially ordered set (poset)*, $D$ is the domain function that interprets terms and variables at each world, $V$ is a labelling function from $\mathbb{P}$ to the power set of the set of all propositional formulas, and $R \in \{\Vdash, \Vdash_c\}$ [10].

For IQC formulas, a Kripke model with the *forcing* relation $\Vdash$ satisfy the following conditions: for all closed formulas $A$ and $B$,

1. for every $q \leqslant p$, $D(p) \subseteq D(q)$ and $V(p) \subseteq V(q)$
2. if $A$ is atomic and $p \Vdash A$, then the terms of $A$ are in $D(p)$
3. $p \nVdash \bot$
4. if $A$ is atomic, then $p \Vdash A$ iff $A \in V(p)$
5. $p \Vdash A \wedge B$ iff $p \Vdash A$ and $p \Vdash B$
6. $p \Vdash A \vee B$ iff $p \Vdash A$ or $p \Vdash B$
7. $p \Vdash \neg A$ iff for every $q \leqslant p$, $q \nVdash A$
8. $p \Vdash A \to B$ iff for every $q \leqslant p$, $q \Vdash A$ implies $q \Vdash B$
9. $p \Vdash \exists x A(x)$ iff for some $a \in D(p)$, $p \Vdash A(a)$
10. $p \Vdash \forall x A(x)$ iff for every $q \leqslant p$ and $a \in D(q)$, $q \Vdash A(a)$

whereas for CQC formulas, a Kripke model with the *classical forcing* relation $\Vdash_c$ also satisfy the clause 1., the analogues of 2., 3., 5., 7. and 10., and the following conditions:

4'. if $A$ is atomic, $p \Vdash_c A$ iff for every $q \leqslant p$, there is $r \leqslant q$ s.t. $A \in V(r)$
6'. $p \Vdash_c A \vee B$ iff for every $q \leqslant p$, there is $r \leqslant q$ s.t. $r \Vdash_c A$ or $q \Vdash_c B$
8'. $p \Vdash_c A \to B$ iff for every $q \leqslant p$, if $q \Vdash_c A$, then there is $r \leqslant q$ s.t. $r \Vdash_c B$
9'. $p \Vdash_c \exists x A(x)$ iff for every $q \leqslant p$, there exists $r \leqslant q$ and $a \in D(r)$, $r \Vdash_c A(a)$

Abusing the notation, $\Vdash$ and $\Vdash_c$ can be lifted as follows:

**Definition 1.** *Let $\mathbb{P}$ be a poset. We note $\mathbb{P} \Vdash \varphi$ if $p \Vdash \varphi$ for every $p \in \mathbb{P}$ (or $1 \Vdash \varphi$ if 1 is the maximal element of $\mathbb{P}$), and similarly for $\mathbb{P} \Vdash_c \varphi$.*

**Definition 2.** *Let $A$ be a set of first-order formulas (axioms).*

*Let $T^i$* (resp. $T^c$) *denote the* intuitionistic theory (resp. classical theory) *of $T$, i.e., the deductive closure of $A$ under* IQC *(resp.* CQC*).*

*We say that $\mathbb{P}$ is* an intuitionistic model (resp. a classical model) of $T$*, denoted* $\mathbb{P} \Vdash T^i$ (resp. $\mathbb{P} \Vdash_c T^c$) *if for every $\varphi \in A$, $\mathbb{P} \Vdash \varphi$ (resp. $\mathbb{P} \Vdash_c \varphi$).*

*Remark 1.* In the case of arithmetic, $T^i$ is HA and $T^c$ is PA.

**Definition 3 (Decidability).** *Relative to a model $\mathbb{P}$ for* $\mathsf{IQC}$ *as above, a formula $\varphi$ is said to be* decidable (over $\mathbb{P}$) *if* $\mathbb{P} \Vdash \varphi \vee \neg\varphi$.

*Remark 2.* Every atomic formula of $\mathsf{HA}$ is decidable [14, 13].

Below are two well-known facts about the forcing relations [10, 11].

**Lemma 1 (Transitivity).** *Let $\Vdash \in \{\Vdash, \Vdash_c\}$. For every formula $\varphi$, if $p \Vdash \varphi$, then for every $q \leqslant p$, $q \Vdash \varphi$.*

**Lemma 2 (Completeness).** $\mathsf{IQC}$ *is sound and complete with respect to $(\mathbb{P}, \leqslant, V, D, \Vdash)$, i.e., for every $\varphi$, $\varphi$ is a theorem of* $\mathsf{IQC}$ *iff for every $\mathbb{P}$, for every $p \in \mathbb{P}$, $p \Vdash \varphi$.*

Note that the completeness of classical forcing w.r.t. $\mathsf{CQC}$ follows from Gödel-Gentzen's double-negation translation:

- $\varphi$ is atomic: $\varphi^{\neg\neg} = \neg\neg\varphi$
- $\varphi = B \circ C$: $\varphi^{\neg\neg} = B^{\neg\neg} \circ C^{\neg\neg}$, for $\circ \in \{\wedge, \rightarrow\}$
- $\varphi = B \vee C$: $\varphi^{\neg\neg} = \neg\neg(B^{\neg\neg}) \vee \neg\neg(C^{\neg\neg})$
- $\varphi = \exists x B(x)$: $\varphi^{\neg\neg} = \neg\neg\exists x(B^{\neg\neg}(x))$
- $\varphi = \forall x B(x)$: $\varphi^{\neg\neg} = \forall x B^{\neg\neg}(x)$

with this translation, we have

$$T^i \vdash_i \varphi^{\neg\neg} \Leftrightarrow T^c \vdash_c \varphi \tag{1}$$

$$\mathbb{P} \Vdash \varphi^{\neg\neg} \Leftrightarrow \mathbb{P} \Vdash_c \varphi \tag{2}$$

(1) has been proved by Gödel [9], and the proof for the equivalence (2) is deferred to the end of Section 3.

**Lemma 3.** *Let $T$ be a theory with the set of axioms $A$. Then for every $\varphi$, $T \vdash_c \varphi$ iff for every $\mathbb{P}$ s.t. $\mathbb{P} \Vdash_c A$, $\mathbb{P} \Vdash_c \varphi$.*

*Proof.* Let $\varphi$ be a theorem of $T^c$. Then there exists (by compactness of first-order logic) a finite subset $P \subseteq A$ s.t. $\vdash_c \bigwedge P \rightarrow \varphi$. Note that the following are equivalent: (i) $\vdash_c \bigwedge P \rightarrow \varphi$; (ii) $\vdash_i (\bigwedge P \rightarrow \varphi)^{\neg\neg}$ by (1); (iii) for every $\mathbb{P}$, $\mathbb{P} \Vdash (\bigwedge P \rightarrow \varphi)^{\neg\neg}$ by completeness; (iv) for every $\mathbb{P}$, $\mathbb{P} \Vdash_c \bigwedge P \rightarrow \varphi$ by (2). Then for every $\mathbb{P}$ s.t. $\mathbb{P} \Vdash A$, as $\mathbb{P} \Vdash \bigwedge P$, $\mathbb{P} \Vdash \varphi$. The converse is proved similarly. □

**Corollary 1 (Completeness of classical forcing).** *For every $\varphi$,* $\mathsf{CQC} \vdash \varphi$ *iff for every $\mathbb{P}$, $\mathbb{P} \Vdash_c \varphi$.*

## 3 Generic sets

Let $(\mathbb{P}, \leqslant)$ be a poset. For every $p, q \in \mathbb{P}$, the *greatest lower bound* $p \wedge q \in \mathbb{P}$ is the greatest element $r$ such that $r \leqslant p$ and $r \leqslant q$, if it exists. Otherwise, $p$ and $q$ are said to be *incompatible*. A *filter* $F \subseteq \mathbb{P}$ on $\mathbb{P}$ is defined as usual: (i) if $p, q \in F$, then $p \wedge q$ exists and belongs to $F$; (ii) if $p \in F$, then for every $q \geqslant p$, $q \in F$ [11].

**Definition 4 (Dense subset).** *A set $D$ is said to be* dense *if for every $p \in \mathbb{P}$, there exists $q \leqslant p$ s.t. $q \in D$.*

**Definition 5 ($\mathbb{P}$-generic filter).** *Let $G \subseteq \mathbb{P}$. $G$ is said to be* $\mathbb{P}$-generic *if $G$ is a filter and $G$ intersects every dense subset of $\mathbb{P}$.*

*For any formula $\varphi$, we define $G \vDash \varphi$ if there exists $p \in G$, $p \Vdash \varphi$.*

**Lemma 4 (Fitting [6]).** *$D_{\varphi \vee \neg\varphi} = \{q : q \Vdash \varphi \vee \neg\varphi\}$ is always dense.*

*Proof.* Let $p \in \mathbb{P}$. Suppose that, for some $q$ s.t. $q \leqslant p$, $q \Vdash \varphi$, then we are done. Otherwise, for every $q$ s.t. $q \leqslant p$, $q \nVdash \varphi$. But then $p \Vdash \neg\varphi$. Therefore, for every $p$, there exists $q \leqslant p$ s.t. $q \Vdash \varphi \vee \neg\varphi$. □

Therefore, $G \vDash \varphi \vee \neg\varphi$. Also, we have $G \nvDash \varphi \wedge \neg\varphi$ for every formula $\varphi$, because it contradicts $G$ being a filter as two elements $p, q \in \mathbb{P}$ s.t. $p \Vdash \varphi$ and $q \Vdash \neg\varphi$ cannot be compatible.

The following lemma shows that every generic filter behaves classically for a restricted class of formulas [5].

**Lemma 5.** *The relation $\vDash$ on a generic filter $G$ can be defined for $\forall$-free formulas as follows:*

- *$G \vDash A$ if there exists $p \in G$ s.t. $p \Vdash A$ for atomic $A$*
- *$G \vDash A \wedge B$ iff $G \vDash A$ and $G \vDash B$*
- *$G \vDash \neg A$ iff $G \nvDash A$*
- *$G \vDash \exists x A(x)$ iff there exists $a \in D(G)$ s.t. $G \vDash A(a)$*

*where $D(G) = \bigcup_{p \in G} D(p)$. $A \vee B$ is defined as $\neg(\neg A \wedge \neg B)$, $A \to B$ is defined as $\neg A \vee B$ and $\forall x A(x)$ is defined as $\neg\exists x \neg A(x)$.*

*Proof.* Let $\vDash'$ denote the original entailment (Definition 5). Show by induction on the complexity of $\varphi$ that $G \vDash \varphi$ iff $G \vDash' \varphi$. □

*Remark 3.* If $G \vDash' \forall x A(x)$, then for every $a \in D(G)$ s.t. $G \vDash' A(a)$. The converse is not true.

Let $\varphi$ and $A$ be formulas, the *double $A$-implication* of $\varphi$ is

$$\varphi^A := (\varphi \to A) \to A$$

The double negation $\neg\neg\varphi$ can then be obtained from $\varphi^A$ by replacing $A$ by $\perp$. The corresponding semantical notions can also be parameterized by $A$.

**Definition 6 ($A$-dense).** *A set $D^A \subseteq \mathbb{P}$ is said to be $A$-*dense *if $D^A = \{p \in \mathbb{P} : p \Vdash \varphi \vee \neg\varphi\}$ holds for some $\varphi \neq A$.*

**Definition 7 ($\mathbb{P}^A$-generic).** *A filter $G^A \subseteq \mathbb{P}$ is said to be $\mathbb{P}^A$-*generic *if $G^A \nvDash A$ and for every $A$-dense subset $D^A$ of $\mathbb{P}$, $G^A \cap D^A \neq \emptyset$.*

*Remark 4.* We may call a $\mathbb{P}^\perp$-generic filter simply *generic*. Note the difference between a $\mathbb{P}^\perp$-generic filter and a $\mathbb{P}$-generic filter in the sense of Definition 5.

*Remark 5.* Every $A$-dense set is also dense by Lemma 4. Therefore, $G^A$ indeed behaves classically whenever $G^A \vDash \neg A$. In particular, Lemma 5 holds for $G^A$ whenever $A$ is decidable.

Let $\mathbb{P}$ be a poset. $\mathbb{P}^A$ is used to denote that $A$ is the parameter for the dense subsets and generic filters over $\mathbb{P}$. We write $p^A$ for an element $p$ of $\mathbb{P}^A$ s.t. $p \nVdash A$. We always use $G^A$ for a $\mathbb{P}^A$-generic filter and $G$ for the generic one. We also use $D_\varphi$ to denote $\{p \in \mathbb{P} : p \Vdash \varphi\}$ for any formula $\varphi$.

The following lemmas are adapted from Cohen's work on forcing [3, 11].

**Definition 8 ($A$-dense below).** *Let $D \subseteq \mathbb{P}^A$ and $p \in \mathbb{P}^A$. $D$ is said to be* $A$-dense below $p$ *if for every $q^A \leqslant p$, there exists $r^A \leqslant q^A$ s.t. $r^A \in D$.*

**Lemma 6.** *Let $G^A$ be $\mathbb{P}^A$-generic and $p^A \in G^A$. Let $\varphi \neq A$. Let $D_\varphi$ is $A$-dense below $p^A$. Then $G^A \cap D_\varphi \neq \emptyset$.*

*Proof.* We first show that if $G^A \cap D_\varphi = \emptyset$, then there exists $q \in G^A$ s.t. for every $r \in D_\varphi$, $r \nleqslant q$.

Since $\varphi \neq A$, $G^A \cap \{p : p \Vdash \varphi \vee \neg\varphi\} \neq \emptyset$ by definition. Assume $G^A \cap D_\varphi = \emptyset$, then $G^A \cap D_{\neg\varphi} \neq \emptyset$. Let $q \in G^A \cap D_{\neg\varphi}$, then $q \Vdash \neg\varphi$ by definition. For every $r \in D_\varphi$, if $r \leqslant q$, then $r \Vdash \neg\varphi$. But $r \Vdash \varphi$ by definition, contradiction.

Now we show $G^A \cap D_\varphi \neq \emptyset$. Assume $G^A \cap D_\varphi = \emptyset$. By the previous statement, let $q \in G^A$ be s.t. for every $r \in D_\varphi$, $r \nleqslant q$. Since $G^A$ is a filter, there exists $q^A = q \wedge p^A$. Note that $q^A \nVdash A$ because $G^A \nvDash A$. As $D_\varphi$ is $A$-dense below $p^A$, there exists $r^A \in D_\varphi$ s.t. $r^A \leqslant q^A$. Since $q^A \leqslant q$ by definition, also $r^A \leqslant q$ by transitivity, contradiction. □

**Lemma 7.** *For every $p \in \mathbb{P}^A$ s.t. $p \Vdash \neg A$, there exists a $\mathbb{P}^A$-generic $G^A$ such that $p \in G^A$.*

*Proof.* We enumerate the set of all non-$A$ formulas as $\varphi_1, \varphi_2, \varphi_3, \ldots$ since it is countable. Let $p_0 = p$. Having defined $p_n$, by Lemma 4, we find $q \leqslant p$ s.t. $q \Vdash \varphi_{n+1} \vee \neg\varphi_{n+1}$. Let $p_{n+1} = q$. One easily verifies that $\{p_n : n < \omega\}$ intersects every $A$-dense set. The filter generated by this set is $\mathbb{P}^A$-generic. □

**Lemma 8.** *Suppose $A$ is decidable. Then for every $p \in \mathbb{P}^A$, $p \Vdash \varphi^A$ iff for every $\mathbb{P}^A$-generic filter $G^A$ with $p \in G^A$, $G^A \vDash \varphi$.*

*Proof.* Suppose $p \Vdash \varphi^A$. If $p \Vdash A$, then $p \Vdash \varphi^A$ always holds and there is no $\mathbb{P}^A$-generic filter going through $p$. So we assume $p \nVdash A$. There are two cases: (i) $A = \varphi$; (ii) $A \neq \varphi$.

(i) Then $p \Vdash A$ as $A^A$ and $A$ are intuitionistically equivalent. It contradicts $p \nVdash A$.

(ii) For every $q \leqslant p$, if for every $r \leqslant q$, $r \Vdash \varphi$ implies $r \Vdash A$, then $q \Vdash A$. Taking the contrapositive, it means that for every $q^A \leqslant p$, there exists $r^A \leqslant q^A$ s.t. $r^A \Vdash \varphi$. In other words, $D_\varphi$ is $A$-dense below $p$. By Lemma 6, given $A \neq \varphi$, for every $G^A$ with $p \in G^A$, $G^A \vDash \varphi$.

Conversely, suppose that for every $\mathbb{P}^A$-generic filter $G^A$ with $p \in G^A$, $G^A \vDash \varphi$. Assume $p \nVdash \varphi^A$. Then there exists $q \leqslant p$ such that $q \Vdash \varphi \to A$ and $q \nVdash A$. Since $A$ is decidable, $q \Vdash \neg A$. By Lemma 7, there is a $G^A$ s.t. $q \in G^A$, which means $G^A \vDash \varphi \to A$. As $p \in G^A$ because $G^A$ is a filter, $G^A \vDash \varphi$ by assumption. Hence $G^A \vDash A$ by modus ponens, contradiction. □

The following statement is an immediate corollary.

**Theorem 1.** *For every $p \in \mathbb{P}$, $p \Vdash \neg\neg\varphi$ iff for every generic filter $G$ with $p \in G$, $G \vDash \varphi$.*

The equivalence (2) in Section 2 then follows by simply replacing $\neg\neg$ by the above formulation in terms of generic filters and then unfolding the definition.

## 4 *A*-translation

Friedman's trick, also known as $A$-translation [5], is a generalization of double-negation translation. The trick consists in replacing every instance of $\bot$ in the derivation of double-negation translations of formulas by $A$, and for every decidable $A$, we can eventually eliminate the double $A$-implication from $(\varphi \to A) \to A$ if $\varphi = A$. In this section, we semantically transform some known double-negation translations into $A$-translations and show that they correctly interpret classical theories into constructive ones.

**Definition 9.** *We use $\varphi_A$ to denote the Friedman's $A$-translation of $\varphi$. More precisely, $\varphi_A$ is the result of replacing every occurrence of every atomic formula $B$ by $B^A$ in $\varphi$.*

**Definition 10.** *Let $\Vdash_i^A$ and $\Vdash_c^A$ denote the following relations:*

$$p \Vdash_i^A \varphi :\Leftrightarrow p \Vdash \varphi^A \qquad\qquad p \Vdash_c^A \varphi :\Leftrightarrow p \Vdash_c \varphi_A$$

The following is a known result by Friedman [7]. But we show a proof using semantical methods.

**Lemma 9.** *Let $A$ be decidable. For every $\Sigma_1$-formula[1] $\varphi$, $\mathbb{P} \Vdash_i^A \varphi$ iff $\mathbb{P} \Vdash_c^A \varphi$.*

*Proof.* We prove by induction a stronger statement: for every $p \in \mathbb{P}$, $p \Vdash_i^A \varphi$ iff $p \Vdash_c^A \varphi$. We only show the nontrivial cases: atomic, $\to$ and $\exists$.

(1) $\varphi$ is atomic.

Note that $p \Vdash_c^A \varphi$ iff $p \Vdash_c \varphi^A$. Suppose $A$ is true ($\mathbb{P} \Vdash A$), then trivially both $p \Vdash_c \varphi^A$ and $p \Vdash \varphi^A$ for every $p$. Otherwise $A$ is false ($\mathbb{P} \Vdash \neg A$) as $A$ is decidable. Then every generic filter is also $\mathbb{P}^A$-generic. So $\mathbb{P} \Vdash \varphi^A$ is equivalent to the fact that for every $\mathbb{P}^A$-generic $G^A$ with $p$, $G^A \vDash \varphi$ by Lemma 8, which is then equivalent to $p \Vdash \neg\neg\varphi$ by Theorem 1, i.e. $p \Vdash_c \varphi$. Note that $p \Vdash_c \varphi \Leftrightarrow p \Vdash_c \varphi \vee A \Leftrightarrow p \Vdash_c \varphi^A$ in the case where $A$ is false.

[1] A $\Sigma_1$-formula is a formula without universal quantifiers.

(2) $\varphi = B \to C$.

Assume $p \Vdash_i^A (B \to C)$. Then for every $G_p^A$ with $p$, $G_p^A \vDash B \to C$. Let $q \leqslant p$ and $q \Vdash_c^A B$. By IH, $q \Vdash_i^A B$, which means that for every $G_q^A$ with $q$, $G_q^A \vDash B$. Note that every $G_q^A$ is also $G_p^A$ since $p \in G_q^A$. So $G_q^A \vDash C$. Thus $q \Vdash_i^A C$. Hence $q \Vdash_c^A C$ by IH. Therefore, $p \Vdash_c^A B \to C$ for $q$ is arbitrary.

Conversely, assume $p \Vdash_c^A B \to C$. So if there is $q \leqslant p$ s.t. $p \Vdash_c^A B$, then there is $r \leqslant q$ s.t. $r \Vdash_c^A C$. Assume for a contradiction that $p \nVdash_i^A B \to C$. Then there exists $G_p^A$ s.t. $G_p^A \nvDash B \to C$. So $G_p^A \vDash B \wedge \neg C$. Then there is $q'' \in G_p^A$ s.t. $q'' \Vdash B \wedge \neg C$, so $q' \Vdash B$ and $q' \Vdash C$ where $q' = p \wedge q''$. Then $q' \Vdash_i^A B$ and $q' \Vdash_i^A \neg C$ and by IH, $q' \Vdash_c^A B$ and $q' \Vdash_c^A \neg C$. By assumption, there is a $r' \leqslant q'$ s.t. $r' \Vdash_c^A C$, but $r' \Vdash_c^A \neg C$ by transitivity. Contradiction.

(3) $\varphi = \exists x B(x)$.

Assume $p \Vdash_i^A \exists x B(x)$. Then for every $G^A$ with $p$, $G^A \vDash B(a)$ for some $a$. Suppose $p \nVdash_c^A \exists x B(x)$. Then for every $q \leqslant p$, $q \nVdash_c^A B(a)$ for every $a \in D(q)$. By IH, $q \nVdash_i^A B(a)$. But $G_p^A$ must intersect some $q \leqslant p$ with $q \Vdash_i^A B(a)$. Contradiction. Hence $p \Vdash_c^A \exists x B(x)$.

Assume $p \Vdash_c^A \exists x B(x)$. So for every $q \leqslant p$, there is $r \leqslant q$ s.t. $r \Vdash_c^A B(a)$ for some $a \in D(r)$. Suppose $B(a) \neq A$. Let $G^A$ be an arbitrary $\mathbb{P}^A$-generic filter with $p$. Let $q \leqslant p$ be in $G^A$, then for some $r \leqslant q$ in $G^A$, $r \Vdash_c^A B(a)$, as otherwise $G^A \vDash \neg B(a)$ and it contradicts the assumption. By IH, $r \Vdash_i^A B(a)$. Thus $G^A \vDash B(a)$ and hence $G^A \vDash \exists x B(x)$. Therefore, $p \Vdash_i^A \exists x B(x)$. If $B(a) = A$, the same conclusion follows trivially. □

**Corollary 2 (Friedman).** *$\mathsf{CQC}$ is $\Pi_2$-conservative over $\mathsf{IQC}$, i.e., $\mathbb{P} \Vdash_c \varphi$ iff $\mathbb{P} \Vdash \varphi$ for any $\Pi_2$-formula[2] $\varphi$.*

This is a classical result, but we have formulated it slightly differently. The proof is deferred to Lemma 14. Also, the semantical interpretation of $A$-translation allows us to generalize other double-negation translations.

**Definition 11 (Kuroda $A$-translation).** *Let $\varphi'$ denote the result of the following transformation of $\varphi$:*

- *If $\varphi$ is atomic, then $\varphi' = \varphi^A$.*
- *If $\varphi = B \circ C$ where $\circ \in \{\vee, \wedge, \to\}$, then $\varphi' = B' \circ C'$.*
- *If $\varphi = \exists x B$, then $\varphi' = \exists x B'(x)$.*
- *If $\varphi = \forall x B$, then $\varphi' = \forall x (B'(x))^A$.*

*The* Kuroda $A$-translation *of $\varphi$ is $\varphi^K = (\varphi')^A$.*

**Lemma 10.** *Let $A$ be decidable. Then $\mathbb{P} \Vdash \varphi^K$ iff $\mathbb{P} \Vdash_c^A \varphi$, where $K$ is the Kuroda $A$-translation defined with respect to $A$.*

*Proof.* We do induction on $\varphi$ to show $\mathbb{P} \Vdash \varphi^K$ iff $\mathbb{P} \Vdash_c^A \varphi$ (as opposed to $p \Vdash \varphi^K$ iff $p \Vdash_c^A \varphi$). We only need to consider the case $\varphi = \forall x B(x)$, for every other case is implied by the proof of Lemma 9.

[2] A $\Pi_2$-formula is a formula of the form $\forall x_1 \ldots \forall x_n \psi$ where $\psi$ is a $\Sigma_1$-formula.

Suppose $\mathbb{P} \Vdash (\forall x(B(x)')^A)^A$. Then for every $\mathbb{P}^A$-generic filter $G^A$, $G^A \vDash \forall x(B(x)')^A$. So for every $a$, $G^A \vDash (B(a)')^A$. Suppose $A$ is false, then every instance of generic filter is $\mathbb{P}^A$-generic. Hence $\mathbb{P} \Vdash (B(a)')^A$. If $A$ is true, then $\mathbb{P} \Vdash (B(a)')^A$ by definition. Thus $\mathbb{P} \Vdash_i^A B(a)'$ and by IH, $\mathbb{P} \Vdash_c^A B(a)$. Therefore, $\mathbb{P} \Vdash_c^A \forall x B(x)$.

Conversely, if $\mathbb{P} \Vdash_c^A \forall x B(x)$, then $\mathbb{P} \Vdash_c^A B(a)$ hence also $\mathbb{P} \Vdash_i^A B(a)'$ by IH for each instance $B(a)$. So $\mathbb{P} \Vdash_i^A \forall x B(x)'$. Since $A$ is decidable, we conclude $\mathbb{P} \Vdash_i^A \forall x(B(x)')^A$, as desired. □

**Definition 12 (Gödel-Gentzen $A$-translation).** *Let $\varphi$ be a formula and the* Gödel-Gentzen $A$-translation *of $\varphi$ (denoted $\varphi^G$) is obtained by inserting double $A$-implications instead of double negations in Gödel-Gentzen's double-negation translation (page 3).*

Adapting the proof for the correctness of Gödel-Gentzen's double-negation translation, we get the correctness of its $A$-variant.

**Lemma 11.** *Let $A$ be decidable. Then $\mathbb{P} \Vdash \varphi^G$ iff $\mathbb{P} \Vdash_c^A \varphi$.*

## 5 Optimal translation

As we can see from the proofs, many aspects of the previous translations could have been optimized. From Kuroda $A$-translation, we see that we do not need to insert double $A$-implication at each occurrence of $\exists, \vee$ and atomic formulas: if there is a successive sequence of these symbols without occurrences of $\forall$ in the middle, we can just add one single double $A$-implication at the occurrence of the main connective. From the Gödel-Gentzen $A$-translation, we notice that there is no need to insert any double $A$-implication when there is no occurrence of $\exists, \vee$ or atomic formulas. In the context of $\mathsf{HA}$, this condition can be weakened to only $\exists$ and $\vee$ as every atomic formula is decidable.

Those different variants of $A$-translation can be combined, so that the specific simplicity of each can complement the complexity of another. Inspired by Gilbert [8], we combine Kuroda $A$-translation and Gödel-Gentzen $A$-translation to obtain the optimal $A$-translation, as defined next.

**Definition 13 (Negative occurrence).** *A connective is said to occur* negatively *in a formula if it occurs in the scope of $\neg$ or $\rightarrow_L$ (left-hand side of an $\rightarrow$). Otherwise, it occurs* positively.

**Definition 14 (Optimal $A$-translation).** *Let $\varphi$ be a formula and $\forall_1, \ldots, \forall_n$ be the occurrences of the universal quantifiers in lexicographic ordering in $\varphi$. Let $\varphi_1, \ldots, \varphi_n$ be the subformulas of $\varphi$ that begin respectively with $\forall_1, \ldots, \forall_n$. For each $\varphi_i$, let $\varphi_{k_1}, \ldots, \varphi_{k_m} \in \{\varphi_{i+1}, \ldots, \varphi_n\}$ be the subformulas of $\varphi_i$ such that for each $k_j$, there are no universal quantifiers between $\forall_i$ and $\forall_{k_j}$, i.e., they are the immediate succeding universal quantifiers of $\forall_i$.*

*For each $\forall_i$, we are interested in whether we need to insert a double $A$-implication between $\forall_i$ and $\forall_{k_1}, \ldots, \forall_{k_m}$. To this end, we only need to take into* *account all connectives occurring between them. We propose the following translation. We must insert a double $A$-implication immediately after $\forall_i$ if the following condition is satisfied at the "fragment" between $\forall_i$ and $\forall_{k_1}, \ldots, \forall_{k_j}$:*

$$\textit{There are positive occurrences of } \vee, \exists \textit{ or atomic formulas.} \tag{3}$$

*More precisely, we define $\varphi^M$ inductively as follows. Let*

$$\varphi_i = \forall x.\varphi'_i(\varphi_{k_1}, \ldots, \varphi_{k_m})$$

*be a universally quantified subformula of $\varphi^M$. Then if $\varphi'_i$ satisfies (3),*

$$\varphi_i^M = \forall x.(\varphi'_i(\varphi_{k_1}^M, \ldots, \varphi_{k_m}^M) \to A) \to A$$

*Otherwise,*

$$\varphi_i^M = \forall x.\varphi'_i(\varphi_{k_1}^M, \ldots, \varphi_{k_m}^M)$$

*Finally, suppose that*

$$\varphi = \varphi'(\varphi_{k'_1}, \ldots, \varphi_{k'_m})$$

*where $\varphi_{k'_1}, \ldots, \varphi_{k'_m}$ are the universally quantified subformulas of $\varphi$ that are not proper subformulas of any other universally quantified subformulas of $\varphi$. Then*

- $\varphi^M = (\varphi'(\varphi_{k'_1}^M, \ldots, \varphi_{k'_m}^M) \to A) \to A$ *if $\varphi'$ satisfies (3)*
- $\varphi^M = \varphi'(\varphi_{k'_1}^M, \ldots, \varphi_{k'_m}^M)$ *otherwise*

*and $\varphi^M$ is said to be the* optimal $A$-translation *of $\varphi$.*

*Remark 6.* In the context of HA, as every atomic formula is decidable, the condition (3) can be weakened as follows: The occurrences of $\vee$ and $\exists$ are positive.

*Example 1.* We provide a step-by-step example of the optimal translation assuming that the atomic formulas are decidable. Let $B, C, D, E, F$ be atomic.

$$(\forall x)(B \wedge (C \to (\forall y(D \vee E))) \vee (\forall z \neg(\exists u F)))$$
$$\rightsquigarrow (\forall x)(B \wedge (C \to (\forall y(D \vee E))) \vee (\forall z \neg(\exists u F)))$$
$$\rightsquigarrow (\forall x)((B \wedge (C \to (\forall y(D \vee E))) \vee (\forall z \neg(\exists u F))) \to A) \to A$$
$$\rightsquigarrow (\forall x)((B \wedge (C \to (\forall y((D \vee E) \to A) \to A)) \vee (\forall z \neg(\exists u F))) \to A) \to A$$

where given a subformula $\varphi_i = \forall x.\varphi'_i(\varphi_{k_1}, \ldots, \varphi_{k_m})$ as before, its red part stands for $\varphi'_i$ and the blue parts stand for $\varphi_{k_1}, \ldots, \varphi_{k_m}$.

**Lemma 12.** *The optimal translation is a valid translation. That is,*

$$\mathbb{P} \Vdash \varphi^M \Leftrightarrow \mathbb{P} \Vdash_c^A \varphi$$

*where $\varphi^M$ is the optimal $A$-translation of $\varphi$ where $A$ is decidable.*

*Proof.* We only sketch the proof, assuming that the atomic formulas are decidable. The proof is by induction. For every inductive step (including the initial case), we only need to consider the "fragment" between a universal quantifier $\forall_i$ and its succeeding universal quantifiers $\forall_{k_1}, \ldots, \forall_{k_m}$ (or the entire $\varphi_i$ in the initial case). If we insert a double $A$-implication immediately after $\forall_i$, it can be rewritten by "pushing through" $\to$ and $\wedge$ to the subformulas within the fragment using the following intuitionistic equivalences:

– $(B \wedge C)^A \equiv B^A \wedge C^A$
– $(B \to C)^A \equiv B \to C^A$

If the double $A$-implication "reaches" $\vee$ or $\exists$, then the Gödel-Gentzen $A$-translation tells us that it cannot be further eliminated. On the other hand, if it reaches one of $\forall_{k_1}, \ldots, \forall_{k_m}$, then it can be eliminated using the following two properties of the universal quantifier:

(i) If $\mathbb{P} \Vdash \varphi^M \Leftrightarrow \mathbb{P} \Vdash_c^A \varphi$, then $\mathbb{P} \Vdash \forall x \varphi^M \Leftrightarrow \mathbb{P} \Vdash_c^A \forall x \varphi$.
(ii) $\mathbb{P} \Vdash (\forall x \varphi^M)^A$ iff $\mathbb{P} \Vdash_c^A \forall x \varphi$.

Finally, the Kuroda $A$-translation tells us that if the double $A$-implications cannot be eliminated within the fragment between $\forall_i$ and $\forall_{k_1}, \ldots, \forall_{k_m}$, then it suffices to insert one single double $A$-implication immediately after $\forall_i$.

**Outline.** More precisely, the induction goes as follows. Note that $\varphi_i, \varphi_i', \varphi_{k_j}$ and $\varphi_{k_j}'$ are defined as in Definition 14.
(a) By induction hypotheses and (i), for every $j \leqslant m$,

$$\mathbb{P} \Vdash (\forall_{k_j} x)(\varphi_{k_j}')^M \text{ iff } \mathbb{P} \Vdash_c^A (\forall_{k_j} x)\varphi_{k_j}';$$

(b) Note that to show that $\mathbb{P} \Vdash \varphi_i^M$ iff $\mathbb{P} \Vdash_c^A \varphi_i$, using (i), it suffices to show

$$\mathbb{P} \Vdash (\varphi_i')^M \text{ iff } \mathbb{P} \Vdash_c^A \varphi_i' \tag{4}$$

(c) If the double $A$-implication reaches $\varphi_{k_j}$, it can be eliminated by (ii) and (a):

$$\mathbb{P} \Vdash ((\varphi_{k_j})^M)^A \text{ iff } \mathbb{P} \Vdash_c^A \varphi_{k_j} \text{ iff } \mathbb{P} \Vdash (\varphi_{k_j})^M$$

(d) Now it remains to show (4). If within the fragment, there are positive occurrences of $\vee$ or $\exists$, then apply the proof for Kuroda $A$-translation, otherwise apply the proof for Gödel-Gentzen $A$-translation. □

## 6 Preservable formulas

In the following, we consider second-order formulas for which the second-order variables cannot be quantified. In other words, such a formula is allowed to have variables ranging over formulas, but the quantifiers can only quantify over first-order variables. In this context, a formula $\varphi(\psi_1, \ldots, \psi_n)$ is said to *have $n$ free placeholders* if $\psi_1, \ldots, \psi_n$ are second-order variables. It is said to be *closed* if the placeholders are absent.

**Definition 15 (Preservable formulas).** *A formula $\varphi$ (potentially having free placeholders) is said to be* preservable in HA *if one of the three following conditions is satisfied:*

- *$\varphi$ is a $\forall$-free formula where every occurrence of $\vee$ and $\exists$ in $\psi$ is negative.*
- *$\varphi = (\forall x)\psi(x)$ where $\psi$ is a preservable formula.*
- *$\varphi = \psi(\varphi_1, \dots, \varphi_n)$ where $\psi, \varphi_1, \dots, \varphi_n$ are all preservable, and $\psi$ has at least $n$ placeholders.*

The following property of preservable formulas can easily be checked by induction on the complexity of formula.

**Lemma 13.** *For every preservable formula $\varphi$ in HA, $\varphi^M = \varphi$.*

Due to the optimal $A$-translation, the provability of the totality of every function definable by a preservable formula is conserved between PA and HA.

**Definition 16 (Decidability of formulas with placeholders).** *Suppose $\varphi$ is an existially quantified formula with n free placeholders. $\varphi$ is said to be $(\varphi_1, \dots, \varphi_n)$-*decidable in HA *if*

$$\mathsf{HA} \vdash \forall x_1 \dots \forall x_m (\varphi(x_1, \dots, x_m, \varphi_1, \dots, \varphi_n) \vee \neg\varphi(x_1, \dots, x_m, \varphi_1, \dots, \varphi_n))$$

**Lemma 14.** *Let $\varphi_1, \dots, \varphi_n$ be closed preservable formulas in HA. Let $\varphi$ be an existially quantified formula with $n$ free placeholders that is $(\varphi_1, \dots, \varphi_n)$-decidable in HA. Then*

$$\mathbb{P} \Vdash_c (\forall x)\varphi(x, \varphi_1, \dots, \varphi_n) \quad \Rightarrow \quad \mathbb{P} \Vdash (\forall x)\varphi(x, \varphi_1, \dots, \varphi_n)$$

*where atomic formulas are decidable on $\mathbb{P}$.*

*Proof.* Suppose $\mathbb{P} \nVdash \forall x \varphi(x, \varphi_1, \dots, \varphi_n)$ and let $a$ be an instance s.t. $\mathbb{P} \nVdash \varphi(a, \varphi_1, \dots, \varphi_n)$. Let $\varphi$ denote $\varphi(a, \varphi_1, \dots, \varphi_n)$. By assumption, $\varphi$ is decidable, so it can be used as the parameter of the optimal $A$-translation. By Lemma 13, $\varphi_1^M = \varphi_1, \dots, \varphi_m^M = \varphi_n$. Then $\varphi^M = \varphi^M(a, \varphi_1^M, \dots, \varphi_n^M)) = (\varphi \to A) \to A$. Substituting $\varphi$ for $A$, the optimal $\varphi$-translation of $\varphi$ is $\varphi^\varphi$. Note that $\mathbb{P} \Vdash \varphi \leftrightarrow \varphi^\varphi$, so $\mathbb{P} \nVdash \varphi^\varphi$. Then we get by Lemma 12 that $\mathbb{P} \nVdash_c^\varphi \varphi$. Since $\varphi$ is false on $\mathbb{P}$ by assumption, $\mathbb{P} \Vdash \varphi_\varphi \leftrightarrow \varphi$, from which also $\mathbb{P} \Vdash_c \varphi_\varphi \leftrightarrow \varphi$. Hence $\mathbb{P} \nVdash_c \varphi$. Therefore, $\mathbb{P} \nVdash_c \forall x \varphi(x, \varphi_1, \dots, \varphi_n)$, as desired. □

**Theorem 2.** *Let $\varphi, \varphi_1, \dots, \varphi_n$ as be in Lemma 14. Then if $(\forall x)\varphi(x, \varphi_1, \dots, \varphi_n)$ is provable in PA, it is also provable in HA.*

*Proof.* We first show that $\mathbb{P} \Vdash \mathsf{HA} \Rightarrow \mathbb{P} \Vdash_c \mathsf{PA}$. Let $\varphi$ be a theorem of PA, then $\varphi^{\neg\neg}$ is a theorem of HA by equivalence (1) (page 3). Then for every HA-model $\mathbb{P}$, also $\mathbb{P} \Vdash \varphi^{\neg\neg}$ by completeness. Hence $\mathbb{P} \Vdash_c \varphi$ by equivalence (2).

By Lemma 3, if $\mathsf{PA} \vdash \varphi$, then for every PA-model $\mathbb{P}$, $\mathbb{P} \Vdash_c \varphi$. Hence also $\mathbb{P} \Vdash \varphi$ by Lemma 14. Since every model of HA is a model of PA, we get that for every HA-model $\mathbb{P}$, $\mathbb{P} \Vdash \varphi$. Therefore, $\mathsf{HA} \vdash \varphi$. □

*Remark 7.* This effectively includes all $\Pi_2$-formulas (Corollary 2).

## 7 Conclusion

In short, this paper extends Friedman's conservativity result by generalizing his $A$-translation to Gilbert's double-negation translation, with the help of Gödel-Gentzen's and Kuroda's double-negation translations. Along the way, we have also developed semantical methods, adapted from Cohen forcing, to interpret these translations. Therefore, the interpretation of classical theories into constructive ones can be viewed as the model transformation from a poset into the collection of all of its generic filters. This potentially gives rise to a new semantical relation based on Kripke models for classically provable formulas, namely the satisfiability on every generic filter. Hence the future work will consist of exploring its connections to classical forcing and thus gain more insight into the relations between classical and constructive theories.

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