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Characterizing logical consequence in many-valued logic*

Emmanuel Chemla† LSCP / ENS

Paul Egré‡ IJN / ENS / SCAS

Benjamin Spector§ IJN / ENS

Abstract

Several definitions of logical consequence have been proposed in many-valued logic, which coincide in the two-valued case, but come apart as soon as three truth values come into play. Those definitions include so-called *pure consequence*, *order-theoretic consequence*, and *mixed consequence*. In this paper, we examine whether those definitions together carve out a natural class of consequence relations. We respond positively by identifying a small set of properties that we see instantiated in those various consequence relations, namely *truth-relationality*, *value-monotonicity*, *validity-coherence*, and a constraint of *bivalence-compliance*, provably replaceable by a structural requisite of *non-triviality*. Our main result is that the class of consequence relations satisfying those properties coincides exactly with the class of mixed consequence relations and their intersections, including pure consequence relations and order-theoretic consequence. We provide an enumeration of the set of those relations in finite many-valued logics of two extreme kinds: those in which truth values are well-ordered and those in which values between 0 and 1 are incomparable.

1 Introduction

Different semantic definitions of logical consequence have been proposed in the area of many-valued logics, which coincide in two-valued classical logic, but come apart as soon as more truth values are admitted. The first of those definitions is the canonical definition in terms of the preservation of designated values from the premises of an argument to the conclusion (see [14, 16, 26, 30, 32]). We call it "pure consequence" (following [5]), to emphasize that the same standard of truth is supposed to be operative in the premises and in the conclusion of an argument. A second definition that has

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been proposed is what we may call "order-theoretic" consequence, according to which an argument is valid provided the conclusion is at least as true as its falsest premise (see [17], [1], [12], [27]). One of the advantages of that definition is that it does not attach truth to a specific standard, but basically says that the conclusion should not meet a worse standard than whatever standard is set for the premises. A third definition we encounter is what we may call "mixed consequence", again following [5]: it says that an argument is valid if and only if, when all premises belong to some designated set of truth values, the conclusion belongs to some possibly distinct designated set (see [13, 18, 20] among others). That definition basically allows for the standard for truth to vary between premises and conclusion, without necessarily requesting a higher standard for the conclusion.

While all three definitions coincide in the two-valued case, it is a remarkable fact that they produce different logics as soon as three values are admitted in the logic. If we call 1 the True, 0 the False, and $\frac{1}{2}$ the Indeterminate, pure consequence already gives two different and dual notions of consequence in that framework, depending on whether only the True is to be preserved from premises to conclusion, or whether the Non-False values are to be preserved. The former notion of validity underlies well-known logics such as Łukasiewicz's three-valued logic L3 (see [2, 16]) and Kleene 's strong logic K3 ([2]), whereas the latter underlies Priest's logic of paradox LP (see [21]) and some conditional logics (viz. [4]). Similarly, validity in terms of mixed consequence yields two dual notions of validity, depending on whether non-false premises should yield a true conclusion, or whether true premises should yield a non-false conclusion. The former notion corresponds to what Malinowski has called q-consequence (see [18]), whereas the latter corresponds to what Frankowski has called p-consequence (see [13], and the review in [7]). Finally, the order-theoretic notion of consequence is equivalent to requesting both the preservation of truth from premises to conclusion, and the preservation of non-falsity from premises to conclusion. The resulting scheme is sometimes called Symmetric Kleene consequence for that matter (and the resulting logic S3, see [9]).

When we look at applications of three-valued logic to various semantic phenomena (presupposition, vagueness, conditionals, paradoxes), another remarkable fact is that those relations of consequence have often been identified or introduced on various independent grounds, usually with different motivations in mind. This concerns not just the pure consequence relations, but also the other two kinds. For example, the p-consequence relation has been proposed in relation to the paradoxes of vagueness ([5, 24, 28, 33]) and to the semantic paradoxes ([6, 22]), but we also find it in the theory of linguistic presuppositions, where it travels under the name of Strawson-entailment ([10, 25], based on [29]).[1] Similarly, the order-theoretic S3 scheme has been considered in relation to the semantic paradoxes ([15]), and it has been suggested independently to account for the semantics of indicative conditionals of natural language ([19]).

We take this to suggest that the three schemes are in a sense natural schemes. But what does "natural" mean exactly? The answer cannot be that such relations are natural merely because

[1] A set of premises Γ is said to Strawson-entail a conclusion A iff whenever all the presuppositions of the premises of Γ and A are jointly satisfied, then the truth of the premises in Γ classically entails the truth of A. (An example of Strawson-entailment is the inference from "John stabbed all of his siblings" to "John stabbed all of his younger siblings" (see [25]): the entailment may be judged problematic if the presupposition of the conclusion that John has younger siblings is not met, even assuming that John has siblings). Strawson-entailment is equivalent in a trivalent framework to saying that whenever all the premises of Γ get the value 1 (which they get only if all their presuppositions get the value 1), then if the presupposition of A gets the value 1, then A gets the value 1. Equivalently: whenever all the premises of Γ get the value 1, either A gets the value $\frac{1}{2}$ (because the presupposition of A does not have value 1) or A has the value 1. This corresponds exactly to the notion of st-validity we review below in section 2.

they conservatively extend classical logic (in the sense of respecting classical consequence when restricted to bivalent valuations, a property we will call *bivalence-compliance*). Considering three-valued logic, for example, many more consequence relations can be defined in that framework, which also coincide with the ones mentioned when viewed through two-valued lenses, but which look gerrymandered. For example, we can define A to be a consequence of Γ when either the truth of the premises in Γ makes A true, or when the non-falsity of the premises in Γ makes A non-false. When all sentences are two-valued, this coincides with two-valued consequence. In the three-valued case, this gives us yet another consequence relation, distinct from the ones mentioned above. Similarly, one could start from any logical consequence relation and add to it a requirement of preservation of 'definedness' (see [25]), namely such that whenever the premises take a value in the set $\{1, 0\}$, so does the conclusion. Such a requirement would of course be vacuous in two-valued logic but it is not necessarily so in many-valued logic. More contrived examples can easily be thought of: we could stipulate that A follows from Γ whenever the truth of all premises of Γ implies the truth of A, and when A gets the value undefined whenever exactly five premises of Γ are undefined. Intuitively, such a definition appears odd, since cardinality considerations ought not to intervene in the definition of logical consequence, but to say this much is not yet to give a criterion for naturalness.

The question we propose to investigate in this paper is exactly this: is there a sense in which the three definitions of logical consequence we mentioned form a natural class? Our proposal is to identify plausible desiderata on a consequence relation in many-valued logic, allowing us to do an exhaustive search of candidates and to narrow down the set of possible consequence relations to a distinguished subset (see [31] for a related project in the case of 4-valued logic). The way we proceed is as follows: we start out by examining the three schemes mentioned above, namely pure consequence, mixed consequence, and order-theoretic consequence, to see what they have in common (section 2). In the three-valued case, those three schemes basically select exactly the logics we mentioned above, but they leave us with a problem of unification: the associated consequence relations have distinct structural properties (not all are transitive or reflexive, for example), and moreover it is simply unclear whether they form a unified class.

Abstracting away from those schemes, we then identify a nontrivial and modest set of constraints that we see instantiated in those various consequence relations (section 3). The constraints in question are fundamentally: for a relation of logical consequence to be *bivalence-compliant*, that is to respect classical consequence over two-valued propositions, to be *truth-relational*, namely to be determined as a relation between sets of truth values and truth values at every point of evaluation (or index), to be *monotonic* relative to the underlying ordering on truth-values, and finally to be *validity-coherent*, namely such as to ensure that if a proposition validly follows from any nonempty premise set, it is a propositional validity. We call "respectable" any consequence relation satisfying those constraints, knowing that bivalence-compliance can be replaced by a structural requisite of *non-triviality*, whose statement does not presuppose classical consequence as a given.

Our main result states that the class of respectable consequence relations in that sense coincides exactly with the class of mixed consequence relations and their intersections, including pure consequence relations and the order-theoretic consequence relation (section 4). This result holds in logics based on any number of truth values. In section 5, we give some results allowing us to enumerate the set of respectable consequence relations in two polar cases: on the one hand for finite-valued logics based on well-ordered truth values, and on the other for finite-valued logics based on a degenerate partial order. What our results show is that the set of respectable consequence relations becomes a very large set when the number of truth-values increases even slightly. While our results provide unification, they still leave us with an abundance of admissible consequence relations, which we may hope to cut down by means of further criteria.

2 Three definitions of logical consequence

In this section we consider the definition of logical consequence in a many-valued setting. We start with a methodological caveat: throughout this paper, we adopt a purely *semantic* perspective, whereby a relation of logical consequence relates *propositions*, instead of sentences of a specific language. We define a proposition in the standard way as a function from indices to truth values. Practically, this means that we study the definition of logical consequence independently of the specification of a set of connectives and its associated semantics. This choice is nonstandard, since consequence relations are usually studied in relation to specific languages, but it affords us more generality and abstraction, in particular it allows us to explore all the possible consequence relations that one might consider in principle.

A limit of our approach is that in many cases there are propositions which cannot be expressed in a given language, and this may significantly modify the properties of the consequence relations that can be defined relative to the expressible propositions of a language. In Appendix A, we discuss how to generalize our approach when the notion of logical consequence is defined in relation to a specific language, including for expressively incomplete languages.[2]

A second assumption we make throughout the paper is that logical consequence is a relation between *sets* of propositions and single propositions. We could define logical consequence in more general terms, between lists of propositions and propositions, so that order and repetition between premises could matter. Likewise, we could define logical consequence to be a relation between sets of propositions and sets of propositions (multi-conclusion case), but we leave this for future work.

2.1 General definitions

The notion of logical consequence is standardly defined on the basis of the notions of truth and falsity. In many-valued logics, the values 1 and 0 are used as best representative of those notions, but there is generally more structure, depending on how close to truth and falsity one takes extra values to be. Formally, this is captured by assuming that a set of truth values comes with an order, in which 1 and 0 stand at the opposite ends. Truth values can be totally ordered or only partially ordered. In what follows we distinguish a class of partial orders which we call *degenerate*, in which basically only the relative positions of 1 and 0 matter. Also, we will informally talk of N-valued logics in the sequel, to refer to either languages interpreted over truth value sets of cardinality N, or to consequence relations based on such sets.

[2]Our semantic approach is even more general than if we stipulated that we confine ourselves to (strongly) functionally complete languages. Consider the most simple case of standard propositional logic with negation, conjunction, a symbol for the True, and infinitely many atoms and with its standard bivalent semantics - in which valuations play the role of what we call indices. The language is (strongly) functionally complete, and there are as many indices as subsets of the integers in that case. However, by compactness the proposition that is true at an index if and only if a finite number of atoms is true at that index cannot be expressed by a sentence of that language (and in fact, not even by a set of sentences of the language). Yet we can include such a proposition as a possible premise or conclusion in our setting.

Definition 2.1 (Truth values and order)**.** A set of truth values $\mathcal{V}$ in many-valued logic is a partially ordered set (by the relation $\leq$) containing 1 and 0, such that 1 is the greatest element, and 0 the least element.

Definition 2.2 (Degenerate partial order)**.** We call a partial order on a set of truth values degenerate whenever all values other than 1 or 0 are pairwise incomparable.

To define propositions, we introduce the notion of an index set. Indices can be viewed as valuations (when a specific language is introduced), or as possible worlds more abstractly. One could naturally require that the index set be infinite (the set of valuations would be infinite for a propositional language with denumerably many atoms), but the only restriction we impose *a priori* on the index set is to not be empty. In practice, the index set need not be infinite for our main results to hold and in most cases, $\mathcal{I}$ can even be a singleton (we will make it explicit when more than one index is needed).

Definition 2.3 (Index set)**.** An index set $\mathcal{I}$ is a nonempty set.

Definition 2.4 (Propositions)**.** A proposition is a function from $\mathcal{I}$ to $\mathcal{V}$, where $\mathcal{I}$ is an index set and $\mathcal{V}$ is a set of truth values. We call $PROP := \mathcal{V}^{\mathcal{I}}$ the set of propositions. Given a proposition P in $PROP$, and any element $i \in \mathcal{I}$, we write $P(i)$ the value of P for i. This is the truth-value of P at index i. Given a set Γ of propositions, $\Gamma(i) := \{P(i); P \in \Gamma\}$.

Definition 2.5 (Constant Propositions)**.** For a truth value x, we note $\overline{x}$ the constant proposition whose value is x at all indices, and for a set of truth values γ we note $\overline{\gamma}$ the set of constant propositions based on the elements of γ. We shall also use the symbols $\top$ and $\bot$ to denote the constant propositions $\overline{1}$ and $\overline{0}$, and we refer to them as "the tautology" and "the contradiction" respectively.

Finally, we define a consequence relation to be any relation between sets of propositions and propositions:

Definition 2.6 (Consequence relations)**.** A consequence relation is a subset of $\mathcal{P}(PROP) \times PROP$. Given a set of propositions Γ, and a proposition C, we write $\Gamma \models C$ to represent that C is a consequence of the propositions in Γ.

Note that this definition is purely extensional and puts no constraints at all on what counts as an admissible consequence relation, or indeed on whether some of these relations specifically deserve to be called logical. We proceed to the examination of such constraints in what follows.

2.2 Three schemes to define logical consequence from truth values

In bivalent logic, the relation of logical consequence is standardly defined in terms of preservation of truth from premises to conclusions. With a more complex set of truth values, a common way to define logical consequence is in terms of preservation of *designated values*. Designated values can be thought of as setting the range of values for which a sentence is either assertible or "true enough" and it is natural to ask for the designated set to be upward closed (see [8, 12]). Let us immediately make this description more precise:

Definition 2.7 (Designated values)**.** A set of *designated values* is a subset $\mathcal{D}$ of $\mathcal{V}$ such that (i) it is not trivial (not empty and not equal to $\mathcal{V}$) and (ii) it is an upset, i.e. if $x \leq y$ and $x \in \mathcal{D}$ then $y \in \mathcal{D}$.

Given that the order may be degenerate, the only substantive constraint that follows is that any set of designated values contains 1 and does not contain 0. We impose condition (i) since including $\emptyset$ or $\mathcal{V}$ can be seen to trivialize consequence relations, as will be illustrated below.

With the notion of designated values in hand, we can replace preservation of truth by preservation of designated values. A second, related way of defining logical consequence relations generalizes the previous scheme in terms of mixed consequence: when all premises take on a designated value, the conclusion must take on a designated value, but the sets of designated values for premises and for conclusions can be different. In other words, this scheme is still about preservation of truth, but what counts as truth may vary from premises to conclusions. The third scheme is an order-theoretic scheme, different from the previous two: the value of the conclusion should never be less than the least value of the premises. That third scheme supposes that any set of values has an infimum (greatest lower bound). Hence, that third scheme is the one that most crucially relies on the existence of a non-degenerate order.

Let us now state formal definitions for these three schemes. Let Γ be a set of propositions, and C a proposition.

Definition 2.8 (Pure consequence)**.** Let $\mathcal{D}$ be a set of designated values. $\Gamma \models_{\mathcal{D}} C$ iff there is no $i \in \mathcal{I}$ such that for every $P \in \Gamma$, $P(i) \in \mathcal{D}$, and $C(i) \notin \mathcal{D}$.

Definition 2.9 (Mixed consequence)**.** Let $\mathcal{D}_p$ and $\mathcal{D}_c$ be two sets of designated values for premises and conclusions, respectively. $\Gamma \models_{\mathcal{D}_p,\mathcal{D}_c} C$ iff there is no $i \in \mathcal{I}$ such that for every $P \in \Gamma$, $P(i) \in \mathcal{D}_p$, and $C(i) \notin \mathcal{D}_c$.

Definition 2.10 (Order-theoretic consequence)**.** $\Gamma \models_{\leq} C$ iff for every $i \in \mathcal{I}$, $\inf(\Gamma(i)) \leq C(i)$, where $\Gamma(i) = \{P(i) | P \in \Gamma\}$.

We will soon discuss the relations between these three schemes. But let us first offer a concrete illustration in three-valued logic.

2.3 The three-valued case

Let us illustrate the above definitions in the three-valued case. Let $\mathcal{V} = \{1, \frac{1}{2}, 0\}$. With three values only, the order is fully determined: 1 is the highest and 0 is the lowest. To define designated values, we can use various standards. When $\mathcal{D} = \{1\}$, call it $\mathcal{S}$ for *strict* truth, we get the standard definition of logical consequence out of the first definition, pure consequence, namely consequence is the preservation of the value 1 from premises to conclusions. When $\mathcal{D} = \{1, \frac{1}{2}\}$, call it $\mathcal{T}$ for *tolerant* truth, we get the dual definition, in terms of preservation of non-falsity. Following the terminology of [5], we call the first relation s-consequence (for strict consequence), and the second t-consequence (for tolerant consequence). Other relations could emerge from that scheme by relaxing some of the constraints of Definition 2.7. If $\mathcal{D}$ could be $\mathcal{V}$ or the empty set, for instance, we would get the universal/trivial relation of consequence u where anything entails anything.

For mixed consequence, all four possible choices of $(\mathcal{D}_p, \mathcal{D}_c)$ have been considered, namely $(\mathcal{S}, \mathcal{S})$, $(\mathcal{T}, \mathcal{T})$, $(\mathcal{S}, \mathcal{T})$, and $(\mathcal{T}, \mathcal{S})$. Following the terminology of [5], we call them ss-consequence, tt-consequence, st-consequence, and ts-consequence. The relations ss and tt obviously coincide with s-consequence and t-consequence, but the other two are new. Many more relations of consequence could be defined out of that scheme if designated values were not upsets. For instance, with $(\mathcal{D}_p, \mathcal{D}_c) = (\mathcal{V}, \emptyset)$, we would get the empty consequence relation: nothing entails anything. With

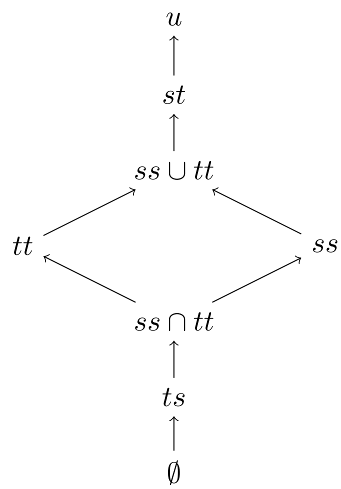

Figure 1. 8 consequence relations

$(\{1,0\},\{1,0\})$, the consequence relation would require that when all the premises take on a classical value, so does the conclusion.[3]

In Figure 1 we list several possible consequence relations. An arrow from one relation to another means that the latter is extensionally more inclusive than the former. Note that we included the intersection of the pure consequence relations ss and tt, as well as their union. Let us discuss some of the properties of these candidate consequence relations.

Fact 2.11. The relations of ss, tt, st, and ts consequence are pairwise distinct. As represented in Figure 1, st is the most inclusive, ts the less inclusive, and ss and tt are between them and are subset-incomparable.

Proof. We refer to [5] for the proof. We note that in Figure 1, we single out $\emptyset$ as a consequence relation to highlight that ts is in general distinct: the constant false proposition ts-entails itself. In [5], however, the language does not contain a constant $\perp$ for the false, hence ts is empty, but that is a contingent feature of the relation. $\square$

The mixed consequence scheme is more general than the standard scheme in terms of designated values, but we can see that it is not fully general.

Fact 2.12. $\leq$-consequence is not one of the mixed consequence relations ss, tt, st, ts.

Fact 2.13. However, $\leq$-consequence is the intersection of ss and tt:

$$\Gamma \models_{\leq} C \quad \text{iff} \quad \Gamma \models_{ss} C \text{ and } \Gamma \models_{tt} C$$

Proofs. These results are direct applications in three-valued logic of the upcoming general Theorems 2.15 and 2.16 in many-valued logics. $\square$

[3] We may call that scheme dd (for 'definedness-to-definedness'). Sharvit in [25] considers the effect of intersecting other more standard consequence relations with it (such as intersecting st with dd).

As a result, pure consequence, mixed consequence and order-theoretic consequence together allow for five distinct nontrivial consequence relations, namely ss, tt, st, ts and $ss \cap tt$. The latter arises from the application of a specific definition, but we note here that it is also definable in terms of the intersection of ss and tt. Pursuing this route, we may find it natural to define a sixth consequence relation as the union of ss and tt. That relation, which we call $ss \cup tt$, is obviously weaker than ss, tt and their intersection, but it is stronger than st:

Theorem 2.14. $\Gamma \models_{ss \cup tt} C$ implies $\Gamma \models_{st} C$, but not conversely, provided $|\mathcal{I}| \geq 2$.

Proof. Assume that $\Gamma \not\models_{st} C$. Then there is an i such that $\Gamma(i) \subseteq \mathcal{S}$ and $C(i) \notin \mathcal{T}$. Since $\mathcal{S} \subseteq \mathcal{T}$, it follows that: $\Gamma \not\models_{ss} C$ (because $\Gamma(i) \subseteq \mathcal{S}$ and $C(i) \notin \mathcal{S}$), $\Gamma \not\models_{tt} C$ (because $\Gamma(i) \subseteq \mathcal{T}$ and $C(i) \notin \mathcal{T}$). Hence, by contraposition, $\Gamma \models_{ss \cup tt} C$ implies $\Gamma \models_{st} C$. The converse does not hold in general however. Let j be an index and let A be the proposition such that $A(j) = \frac{1}{2}$, and $A(i) = 1$ for all $i \neq j$ (that such i and j exist is guaranteed by the assumption on the cardinality of $\mathcal{I}$). Let B be the proposition such that $B(j) = 0$ and $B(i) = \frac{1}{2}$ for every $i \neq j$. Clearly, $A \not\models_{tt} B$, since $A(j) \neq 0$ and $B(j) = 0$, and likewise $A \not\models_{ss} B$, since for some j $A(j) = 1$ and $B(j) < 1$. So, $A \not\models_{ss \cup tt} B$. But $A \models_{st} B$, since $A(j) \notin \mathcal{S}$, and for any $i \neq j$, $B(i) \in \mathcal{T}$. $\square$

Figure 1 represents the closure of the four mixed consequence relations we distinguished under union and intersection, with also the universal and the empty consequence relations. In light of that figure, we may find natural to elect the six non-trivial consequence relations as "natural" relations. However, a principled argument to do so is missing. In particular, we need an argument that the "naturalness" of two consequence relations is preserved under union.

The proof of the previous theorem suggests an argument against viewing $ss \cup tt$ as a 'natural' consequence relation. The two propositions A and B used to show that $ss \cup tt$ is different from st are such that $A \not\models_{ss \cup tt} B$ and yet such that for every index i, the constant propositions $\overline{A(i)}$ and $\overline{B(i)}$ are such that $\overline{A(i)} \models_{ss \cup tt} \overline{B(i)}$ (if u is a truth-value, we denote with $\overline{u}$ the constant proposition that maps every index to u). In other words, for $ss \cup tt$, whether the logical consequence relation holds between two propositions cannot be determined by checking whether it holds at each index independently of the others. On the contrary, it is immediate to see that the other five relations are all such that $A \models B$ if and only if for every i, $\overline{A(i)} \models \overline{B(i)}$. We will refer to this fact by saying that $ss \cup tt$ is not truth-relational (see section 3.2), which makes it special among the possible relations we have discussed so far in three-valued logics. The lack of truth-relationality of $ss \cup tt$ is for us an indication that the latter is not on a par with the others. But this also suggests that we might seek a characterization of those five remaining relations in terms of truth-relationality and other conditions (since obviously, more possible relations than those five are truth-relational). Our goal in what follows will be exactly to identify such conditions.

2.4 General comparison of the three schemes

In three-valued logic we were able to investigate the relations between the three schemes. In this section, we demonstrate that these results generalize to many-valued logic. Obviously, pure consequence is a particular case of mixed consequence in which $\mathcal{D}_p = \mathcal{D}_c$, but the relation of order-theoretic consequence with the other two schemes is less obvious. Specifically:

Theorem 2.15. In many-valued logics (with more than two values), $\leq$-consequence is not a mixed consequence relation.

Proof. Suppose $\mathcal{D}_p$ and $\mathcal{D}_c$ are sets of designated values such that the mixed consequence relation $\models_{\mathcal{D}_p,\mathcal{D}_c}$ based on them is the order-theoretic relation. Let x be in $\mathcal{D}_c$. Then the constant proposition $\overline{x}$ is a validity (all propositions $(\mathcal{D}_p, \mathcal{D}_c)$-entail it), so x has to have the highest possible value, 1, (so that all propositions $\leq$-entail $\overline{x}$, in particular the constant proposition $\overline{1}$ itself should entail it). Hence, $\mathcal{D}_c = \{1\}$. Now let y be an element of $\mathcal{D}_p$. Since all constant propositions $\leq$-entail themselves, it has to be that y is also in $\mathcal{D}_c$, so $\mathcal{D}_p \subseteq \mathcal{D}_c$, and therefore $\mathcal{D}_p = \mathcal{D}_c = \{1\}$. Finally, choose z different from 0 and 1. The constant proposition $\overline{z}$ does not $\leq$-entail the constant proposition $\overline{0}$, even though it $(\{1\}, \{1\})$-entails it. Contradiction. □

Nonetheless, as we saw in three-valued logics, the connection between order-theoretic consequence and the other schemes does exist if we accept to add *intersection* as a way of producing consequence relations: when there is an appropriate order over the set of truth values to define the order-theoretic consequence, then this order-theoretic consequence can be obtained as the intersection of pure consequence relations (see [12] for statement of this result).

Theorem 2.16. The order-theoretic relation is equivalent to the intersection of all pure consequence relations based on upsets containing their infimum.

Proof. Let $\models_{\leq}$ be the order-theoretic relation. $\Gamma \models_{\leq} C$ iff for every $i \in \mathcal{I}$, $\inf(\Gamma(i)) \leq C(i)$
iff for every $i \in \mathcal{I}$, for every upset $\mathcal{D}$ containing its infimum, if $\Gamma(i) \subseteq \mathcal{D}$ then $C(i) \in \mathcal{D}$
iff for every upset $\mathcal{D}$ containing its infimum, there is no $i \in \mathcal{I}$, $\Gamma(i) \subseteq \mathcal{D}$ and $C(i) \notin \mathcal{D}$
iff for every upset $\mathcal{D}$ containing its infimum, $\Gamma \models_{\mathcal{D}} C$. □

To sum up, in many-valued logic the order-theoretic consequence relation is an intersection of pure consequence relations, which themselves are a special case of mixed consequence relations. If we accept that intersection is a natural operation over consequence relations, then the disparity between the three schemes is only apparent and they can all be obtained from mixed consequence relations. However, this raises two issues. First, in the general case full closure under intersection might possibly include more relations than simply the ones we found in the cases considered so far, including intersections of mixed consequence relations different from the others and from the order-theoretic consequence relation. Secondly, we need to motivate the introduction of intersection (as opposed to other closure operations, such as union) as a way to extend the schemes. We saw previously that truth-relationality is preserved under intersection, and not under union. This feature will play a central role in what follows, but what we need is to identify more general principles of the same kind.

3 Respectable consequence relations

In this section we state a list of properties that we use to define a class of consequence relations we call *respectable*. We saw in the last section that in the three-valued case the relation $ss \cup tt$ fails to be truth-relational, unlike other relations depicted in Figure 1. Beside truth-relationality, we here identify other properties of those relations that they have in common, and which we shall use toward our main characterization result.

3.1 Bivalence-Compliance

The first desideratum we put on a respectable consequence relation is to be conservatively extending classical two-valued consequence in the sense of coinciding with two-valued consequence when applied to two-valued propositions:

Definition 3.1 (Bivalent propositions and classical two-valued consequence)**.** A proposition is called *bivalent* if it takes values solely in $\{0, 1\}$. Given a set Γ of bivalent propositions, and a bivalent proposition C, we write $\Gamma \models_c C$ to mean that C is a classical two-valued consequence of Γ, that is, for every i, C takes the value 1 at i if all of the premises in the set Γ take the value 1 at i.

Definition 3.2. A consequence relation $\models$ is *bivalence-compliant* if for any set of bivalent premises Γ and any bivalent conclusion C,

$$\Gamma \models C \quad \text{iff} \quad \Gamma \models_c C$$

We emphasize that bivalence-compliance is only a very weak requirement: to say that a relation is bivalence-compliant does not imply that it is classical in general, but only that it treats classical propositions classically. A drawback of that requirement, arguably, is that it appears to presuppose the notion of classical consequence, instead of deriving it. Below, however, we will see that we can state a distinct requisit that does not have that feature, but which is provably equivalent, given the other constraints we lay out in our approach.

3.2 Truth-relationality

Truth-relationality states that whether a conclusion follows from a set of premises can be assessed by looking at the truth values of the relevant propositions at each index separately (through some relation $\triangleleft$ active at the level of truth-values, rather than propositions). We already illustrated the property in the three-valued case and saw in what sense it sets the union $ss \cup tt$ of two consequence relations apart from the others. We now state the definition in a more general setting:[4]

Definition 3.3 (Truth-relationality)**.** A consequence relation $\models$ is *truth-relational* if there is a relation $\triangleleft$ between elements of $\mathcal{P}(\mathcal{V})$ and elements of $\mathcal{V}$ such that for all sets of propositions Γ and all propositions C, $\Gamma \models C$ iff $\forall i \;\; \Gamma(i) \triangleleft C(i)$.

A truth-relational consequence relation is fully determined by its behavior with respect to *constant* propositions, and this property could serve as the basis for an equivalent definition of truth-relationality. This is what the following theorem expresses.

Theorem 3.4. A consequence relation $\models$ is truth-relational if and only if, for all sets of propositions Γ and all propositions C, $\quad \Gamma \models C \quad$ iff $\quad \forall i \; \overline{\Gamma(i)} \models \overline{C(i)}$.

[4]Truth-relationality as we state it is a distinct notion from the notion of being characterizable by a matrix (in the sense of [32]). To say that a consequence relation is characterized by a matrix implies for it to be characterizable in terms of *pure* consequence, that is in terms of the preservation of a single set of designated values (see [32]). The intersection of two consequence relations each characterized by a matrix may not be characterizable by a matrix, but it follows from our definition that the intersection of two truth-relational consequence relations is always truth-relational (see Theorem 4.3). In the terms of our paper, an example is $ss \cap tt$: it is an intersection of two pure consequence relations (i.e. both ss and tt are characterizable by a matrix), but not itself a pure consequence relation (see Theorem 2.15). Hence it is not characterizable by a matrix, but it is nevertheless truth-relational in our sense.

Proof. We need to prove the equivalence between the following two statements:

(a) There is a relation $\triangleleft$ between elements of $\mathcal{P}(\mathcal{V})$ and elements of $\mathcal{V}$ such that:
For all Γ, C, $\quad \Gamma \models C \quad$ iff $\quad \forall i \;\; \{P(i) : P \in \Gamma\} \triangleleft C(i)$.

(b) For all Γ, C, $\quad \Gamma \models C \quad$ iff $\quad \forall i \;\; \{\overline{P(i)} | P \in \Gamma\} \models \overline{C(i)}$.

Proof that (a) entails (b). Let us assume (a). Let $\triangleleft$ be a relation that satisfies the existential claim of (a). We have, for all Γ and C:

$\Gamma \models C \quad$ iff $\quad \forall i \;\; \{P(i) | P \in \Gamma\} \triangleleft C(i)$ (by (a)).
iff $\quad \forall i \forall j \;\; \{\overline{P(i)}(j) | P \in \Gamma\} \triangleleft \overline{C(i)}(j)$ (for any proposition X, $\overline{X(i)}(j) = X(i)$).
iff $\quad \forall i \;\; \{\overline{P(i)} | P \in \Gamma\} \models \overline{C(i)}$ (by (a) applied to '$\forall j \;\; \{\overline{P(i)}(j) | P \in \Gamma\} \triangleleft \overline{C(i)}(j)$').

Proof that (b) entails (a). Let us assume (b) and define the relation $\triangleleft$ as follows: for E is a set of truth-values and v a truth-value, $E \triangleleft v$ iff $\{\overline{u} | u \in E\} \models \overline{v}$. It is straightforward that $\triangleleft$ satisfies the existential claim of (a). Indeed, for all Γ and C:
$\Gamma \models C \quad$ iff $\quad \forall i \;\; \{\overline{P(i)} | P \in \Gamma\} \models \overline{C(i)}$ (by (b))
iff $\quad \forall i \;\; \{P(i) | P \in \Gamma\} \triangleleft C(i)$ (by definition of $\triangleleft$). □

Corollary 3.5. If a consequence relation $\models$ is truth-relational, then the relation $\triangleleft$ that makes it truth-relational is unique.

Proof. By Theorem 3.4, the relation $\triangleleft$ which makes $\models$ truth-relational (cf. Definition 3.3) is fully determined by the behavior of $\models$ with respect to constant propositions. □

In Appendix A, we note that this uniqueness is no longer guaranteed when consequence relations are defined over the propositions which are expressible in some given interpreted language, if these propositions do not include all the constant propositions.

3.3 Value-monotonicity

Recall that truth values come with an order, be it degenerate or not. That order gives sense to the mapping between those values and the pre-theoretic notions of truth and falsity, see section 2.1. What we call value-monotonicity materializes the meaning of this order into the consequence relation, by saying that propositions that are closer to truth are entailed by more propositions, and propositions that are closer to falsity entail more propositions. To achieve this result, we must first infer from the order of truth values a corresponding notion of what counts as a stronger proposition or a stronger set of propositions:

Definition 3.6 (Order on propositions and sets of propositions)**.** Based on the order $\leq$ on truth values, we define orders on propositions and sets of propositions (note that 'stronger' is not to be understood in the sense of a strict order):

- A proposition P_2 is stronger than another proposition P_1, noted $P_2 \leq P_1$, iff $\forall i, P_2(i) \leq P_1(i)$.
- A set of propositions Γ_2 is stronger than another set of propositions Γ_1, noted $\Gamma_2 \leq \Gamma_1$, iff $\forall P_1 \in \Gamma_1, \exists P_2 \in \Gamma_2$ such that $P_2 \leq P_1$.

With these definitions in hand, we can define value-monotonicity, as the announced property that stronger premise sets should entail more propositions, and that weaker conclusions should be entailed by more premise sets:

Definition 3.7 (Value-monotonicity)**.** A consequence relation is *value-monotonic* if for any sets of premises (Γ_1, Γ_2), with Γ_2 stronger than Γ_1, and conclusions (C_1, C_2), with C_2 weaker than C_1:

$$\text{If } \Gamma_1 \models C_1, \text{ then } \Gamma_2 \models C_2$$

It is useful to realize that if the order over truth values is minimal, value-monotonicity is also minimal in that it gives their expected, extreme roles to the constant propositions $\overline{0}$ and $\overline{1}$. To see this, consider the case where truth-values come with a degenerate order (see Definition 2.2), and let us focus on what happens on constant propositions for a truth-relational consequence relation (see Theorem 3.4). The part of value-monotonicity concerned with the replacement of premise sets amounts to saying that if the tautology entails some proposition P, any proposition entails that same proposition P; similarly, the part concerning replacement of conclusions amounts to saying that if some proposition P entails the contradiction, that proposition P entails any other proposition. In that case, value-monotonicity is thus minimal in the sense that it imposes no other constraint on propositions with intermediate values. When the order on truth values is no longer degenerate, value-monotonicity lifts it into corresponding relations between premises and conclusions, allowing for a finer-grained control on relations between propositions taking intermediate values.

It is also useful to compare this notion of value-monotonicity with the Tarskian notion of monotonicity (see [30]'s Axiom 4, from which the current definition follows):

Definition 3.8 ((Tarskian) Monotonicity)**.** A consequence relation $\models$ is *monotonic* if for any sets of premises Γ and Γ' and any proposition C,

$$\text{If } \Gamma \models C, \text{ then } \Gamma \cup \Gamma' \models C.$$

Tarskian monotonicity requires that at least as many consequences are obtained when premises are added. We can formally demonstrate the following relation between Tarskian monotonicity and value-monotonicity:

Fact 3.9. Value-monotonicity entails monotonicity in the Tarskian sense.

Proof. Assume that $\Gamma \models C$ and let Γ' be a set of propositions. Clearly, for all $P_1 \in \Gamma$ there is $P_2 \in \Gamma \cup \Gamma'$ such that $P_1 \geq P_2$, i.e. $\Gamma \cup \Gamma' \geq \Gamma$. Thus value-monotonicity guarantees that $\Gamma \cup \Gamma' \models C$. □

Tarsksian monotonicity can be seen as a particular application of the exact same definition as the one for value-monotonicity, only with a different ordering relation over sets of propositions: while value-monotonicity as we have it relies on the ordering relation defined in Definition 3.6, Tarskian monotonicity relies on the inclusion ordering relation. From there, both notions state that stronger sets of premises give rise to more consequences. Tarskian monotonicity, however, does not rely on the ordering relation between truth values to define a notion of strength over premise sets, and it can thus be stated in structural terms. This may be seen as an advantage of the Tarskian definition. However, the inclusion relation is mostly silent at the level of propositions; for instance it does not cash out the intuition that $\perp$ is stronger than $\top$. The ordering relation behind value-monotonicity, on the other hand, makes distinctions at the level of propositions as well.[5] Because of that, value-monotonicity can state that the consequence relation is preserved not only when premises are strengthened, but also when conclusions (single propositions, not sets) are weakened, a feature we cannot obtain in purely structural terms without introducing independent machinery to compare propositions (or singletons of propositions). Value-monotonicity does so by taking advantage of the ordering at the level of truth values.

3.4 Validity-coherence

We state a fourth constraint on consequence relations, which concerns the relation between valid arguments based on empty sets of premises (*propositional validities*) and valid arguments based on nonempty sets of premises. The constraint, which we call *validity-coherence*, ensures that a proposition that follows validly from every nonempty premise set ought to be a propositional validity, that is it should also follow validly from the empty set of premises:

Definition 3.10 (Validity-coherence)**.** A consequence relation is *validity-coherent* if for every proposition C:

$$\text{If } \forall \Gamma \neq \emptyset, \Gamma \models C, \text{ then} \models C$$

Validity-coherence is a property of a number of logical systems, and it is natural to assume it.[6] It is worth noting that validity coherence can be seen as the converse of Tarskian monotonicity (Definition 3.8):

Fact 3.11. Validity-coherence is equivalent to the property that for every Γ and C:

$$\text{If } \forall \Gamma' \neq \emptyset, \Gamma \cup \Gamma' \models C, \text{ then } \Gamma \models C.$$

Proof. If Γ is not empty, this statement is trivially true: the right-hand side is a particular instantiation of the left-hand side (pick $\Gamma' \subseteq \Gamma$). The only substantial part of it is thus concerned with $\Gamma = \emptyset$, which corresponds to validity-coherence. □

Finally, let us note that together with value-monotonicity, validity-coherence ensures that validities are fully determined by the relation with nonempty sets of premises, in a meaningful way:

Fact 3.12. If a consequence relation is value-monotonic and validity-coherent, then the set of propositional validities is fully determined by the set of nonempty-premise relations (and also by the set of single-premise relations). Formally, for all propositions C:

$$\models C \text{ iff } \forall P, P \models C \text{ iff } \forall \Gamma \neq \emptyset, \Gamma \models C.$$

[5] Definition 3.6 seems to be offering two independent types of ordering relations: one between propositions and one between sets of propositions. But these definitions collapse if we assimilate propositions with singletons of propositions. The one line version of the underlying relation is: A set of propositions Γ_2 is stronger than another set of propositions Γ_1, noted $\Gamma_2 \leq \Gamma_1$, iff $\forall P_1 \in \Gamma_1, \exists P_2 \in \Gamma_2$ such that $\forall i, P_2(i) \leq P_1(i)$.

[6] For instance, in a logic over a language with negation, and satisfying the rule of Reasoning by Cases, namely such that $P \models C$ and $\neg P \models C$ imply that $\models C$. This constraint suffices to ensure validity-coherence. More broadly, validity-coherence imposes that Reasoning by Cases holds if one considers as the cases all the properties that can be expressed in the language. Independently of that comparison with Reasoning by Cases, validity-coherence is also natural in a language that can express the tautology, because it is natural to request that if $\top \models C$, then $\models C$.

Proof. Value-monotonicity ensures the left-to-right implications (preservation of consequence from smaller to larger sets of premises). The right-to-left implications are either obvious or equivalent to validity-coherence. □

Anticipating a bit, we can assess exactly what role validity-coherence plays for us. Below, we define a class of respectable consequence relations, satisfying a number of properties, including validity-coherence. If we were to drop validity-coherence from that list, we would obtain almost the same class of consequence relations: it would contain all the respectable consequence relations, plus some more, which would always be almost entirely similar except that their set of propositional validities may be smaller, somewhat arbitrarily, precisely due to the absence of a principled connection between propositional validities and valid arguments based on nonempty premise sets. In the Appendix, however, we show that validity-coherence can fail in languages that do not express all constant propositions.

3.5 Respectable relations and a comparison with Tarski's conditions

We have now identified various properties a relation between a set of premises and a conclusion could have. We collect these properties in the following definition.

Definition 3.13 (Respectability)**.** A consequence relation is *respectable* if it is bivalence-compliant, truth-relational, value-monotonic and validity-coherent.

Our goals should now be highly reminiscent of Tarski's program in [30], who however proposed a different set of constraints to define *logical* consequence, namely: reflexivity, transitivity and monotonicity. We have already defined monotonicity, the other two properties can be stated as follows.

Definition 3.14 (Reflexivity)**.** A consequence relation $\models$ is *reflexive* if for any set of premises Γ and proposition A,

$$\Gamma, A \models A.$$

Definition 3.15 (Transitivity)**.** A consequence relation $\models$ is *transitive* if for any Γ, A, B and C:

$$\Gamma, A \models B \text{ and } \Gamma, B \models C \quad \text{implies} \quad \Gamma, A \models C.$$

Importantly, if all respectable relations are monotonic (because they are value-monotonic, see Fact 3.9), not all are logical in Tarski's sense because they may not satisfy the other properties. In particular, note that st is not transitive and ts is not reflexive (over functionally complete languages), even though they are respectable. However, those relations have been identified as meaningful and legitimate forms of consequence in relation to various phenomena (see again [18, 23, 33]), and they will be proven to be respectable. Here, we depart from the Tarskian perspective and do not impose either reflexivity or transitivity on what we shall call respectable consequence relations.

We still focus attention to monotonic logics however, even though monotonicity too has been challenged as a property a logic should have, in particular in relation to ordinary reasoning. The typical nonmonotonic logics we can think of are not defined in an N-valued framework, but we can define N-valued truth-relational consequence relations that fail to be monotonic (see below). In what follows, we will call "respectable" a consequence relation only if it is monotonic, but obviously, we use "respectable" as a term of art, not to mean that nonmonotonic consequence relations are not to be given due consideration.

3.6 Independence of the properties

We show that the four defining properties of the notion of respectability are independent of each other, that is:

Theorem 3.16. None of the defining properties of a respectable consequence relation follows from a subset of the others.

Proof. We exhibit examples of consequence relations that satisfy any three of the four properties without the fourth. Whenever necessary, we assume that $\mathcal{I}$ has more than one element and that the set of truth values contains at least 3 values.

All properties but truth-relationality are satisfied by $ss \cup tt$, assuming that $\mathcal{I}$ is of cardinality at least 2. Fact 2.14 demonstrates that $ss \cup tt$ does not satisfy Theorem 3.4, hence it is not truth-relational. It is bivalence-compliant, value-monotone and validity-coherent, because each of ss and tt has those properties (see Theorem 4.2 for a general result), and those properties are closed under union. For example, in the case of validity-coherence: assume that $\forall \Gamma \neq \emptyset, \Gamma \models_{ss \cup tt} C$. Then $\top \models_{ss \cup tt} C$. Hence, either $\top \models_{ss} C$ or $\top \models_{tt} C$. Evidently, $\models_{tt} C$ and therefore $\models_{ss \cup tt} C$.

All properties but value-monotonicity are satisfied by the consequence relation $\models$ defined by $\Gamma \models C$ iff Γ and C only consist of bivalent propositions, and $\Gamma \models_c C$. This is not monotonic even in the Tarskian sense, since for example $\bot \models \bot$, but $\bot, P \not\models \bot$ for any nonbivalent proposition P. This is clearly bivalence-compliant. This is vacuously validity-coherent, because no proposition is entailed by all propositions (nonbivalent propositions entail nothing). This is made truth-relational by the relation that holds of (γ, c) iff γ and c consist of only classical truth values and $\overline{\gamma} \models_c \overline{c}$.

All properties but bivalence-compliance are satisfied by a number of consequence relations: (i) the relation such that everything entails everything, (ii) the relation such that nothing entails anything, (iii) any respectable $\models$ modified so that nothing entails a proposition which ever takes the value 0 (formally: $\Gamma \models' C$ iff $\Gamma \models C$ and $0 \notin C(\mathcal{I})$), (iv) any respectable $\models$ modified such that no proposition which ever takes the value 1 contributes to an entailment of anything (formally: $\Gamma \models' C$ iff $\tilde{\Gamma} \models C$, where $\tilde{\Gamma} = \{P \in \Gamma : 1 \in P(\mathcal{I})\}$).

All properties but validity-coherence are satisfied by the consequence relation which is exactly like st for nonempty premise sets, but which contains only $\top$ as a validity. $\square$

3.7 Non-triviality as an alternative to bivalence-compliance

To conclude this section, we give an alternative characterization of the notion of bivalence-compliance relative to the three other properties, in terms of non-triviality. Non-triviality effectively plays the exact same role as bivalence compliance (except for non-maximally expressible languages, as discussed in Appendix A). However, non-triviality is a purely structural constraint, unlike bivalence-compliance.

Definition 3.17 (Non-triviality)**.** A consequence relation $\models$ is non-trivial if: (i) $\models$ is not the universal relation (i.e. $\exists(\Gamma, C)$ such that $\Gamma \not\models C$), (ii) every proposition must entail some proposition, and (iii) every proposition must be entailed by some proposition.

Theorem 3.18. A consequence relation is respectable iff it is truth-relational, value-monotonic, validity-coherent and non-trivial (i.e. bivalence-compliance can be replaced by non-triviality).

Proof.

- Bivalence-compliance clearly implies (i) of the non-triviality condition, (ii) follows from the fact that $\top$ is entailed by all propositions (simply by monotonicity and $\models \top$) and (iii) follows from the fact that $\bot$ entails any proposition (given that it entails $\bot$, the strongest of the propositions in the sense of value-monotonicity).

- Conversely, assume that a consequence relation $\models$ is truth-relational, value-monotonic, validity-coherent, and non-trivial and let us prove that it is bivalence-compliant.

  Because the relation is not universal, one can find Γ and C such that $\Gamma \not\models C$. Since $\bot \leq C$ for every proposition C, it follows by value-monotonicity that $\Gamma \not\models \bot$. (a) If Γ is not empty, then it follows that $\top \not\models \bot$ by value-monotonicity. (b) If Γ is empty, then $\not\models \bot$, but by validity-coherence it follows that there is a non-empty Γ' such that $\Gamma' \not\models \bot$ and we conclude as in (a) that $\top \not\models \bot$.

  We can similarly prove that $\not\models \bot$ (by value-monotonicity), that $\top \models \top$ (otherwise $\top$ would not entail any proposition, by value-monotonicity), that $\models \top$ (by validity-coherence), that $\bot \models \bot$ (otherwise $\bot$ would not be entailed by any proposition, by value-monotonicity) and that $\bot \models \top$ (again by value-monotonicity).

  The above establishes bivalence-compliance restricted to (a) single or empty premise sets and (b) constant propositions. But regarding (a), the multi-premise cases are fully determined by monotonicity (the only multiple-premise set made of constant, bivalent propositions is $\{\top, \bot\}$ and it entails both $\top$ and $\bot$ based on monotonicity and the fact that $\bot$ alone does). And regarding (b), the extension to non-constant propositions is constrained appropriately by truth-relationality. □

We therefore obtain an equivalent definition of a respectable consequence relation, by replacing bivalence-compliance with non-triviality. An advantage of non-triviality over bivalence-compliance, given the other constraints, is that we can define a respectable consequence relation without invoking classical consequence. Instead, we can prove that two-valued classical consequence (that is, *ss*-consequence, over $\mathcal{V} = \{1, 0\}$) just is the unique two-valued respectable consequence relation in the sense of those four conditions, without begging the question of what makes classical consequence special:

Theorem 3.19. Classical consequence in two-valued logic is the unique bivalent consequence relation which is truth-relational, value-monotonic, validity-coherent and non-trivial.

Proof. This follows from the equivalent definition of respectability (in terms of truth-relationality, value-monotonicity, validity-coherence and non-triviality), from our upcoming general characterization Theorem 4.1 of respectable relations, and from the fact that classical consequence is the only mixed consequence relation in bivalent logic. □

We further note that this provides a *minimal* characterization of classical consequence, in the sense that these four properties are independent even in bivalent logic (parallel to Fact 3.16):

Fact 3.20. None of the properties among truth-relationality, value-monotonicity, validity-coherence and non-triviality follows from a subset of the others. This holds in bivalent logic and higher (assuming that $|\mathcal{I}| \geq 2$).

Proof. As in the proof of Fact 3.16, we exhibit examples of consequence relations that satisfy any three of the four properties without the fourth, here all in bivalent logic.

All properties but truth-relationality are satisfied by the relation $\models^{i_0}$ which is defined as: $\Gamma \models^{i_0} C$ iff $\Gamma(i_0) \models_C C(i_0)$, for i_0 some specific element of $\mathcal{I}$.

All properties but value-monotonicity are satisfied by the relation which is exactly like classical consequence except that the role of 0 and 1 are exchanged (in short: $\Gamma \models C$ iff $\{\neg P : P \in \Gamma\} \models_C \neg C$).

All properties but non-triviality are satisfied by, e.g., the universal relation.

All properties but validity-coherence are satisfied by the consequence relation which is exactly like classical consequence but has no propositional validity. □

We mostly use the definition of respectability based on bivalence-compliance in what follows, but it is important to bear in mind the equivalent characterization in terms of nontriviality. The latter, in particular, appears more general toward giving a fully structural characterization of the class of respectable relations. This program, however, lies beyond the scope of the present paper.[7]

4 Respectable consequence relations and mixed consequence relations

In section 3 we set up a general definition of what a consequence relation may be and what fundamental properties it must satisfy, leading us to the notion of a *respectable* consequence relation (Definition 3.13). In this section, we relate the class of such respectable consequence relations to the schemes used to define logical consequence in section 2. We immediately state our main theorem, which we prove thereafter in the next two subsections:

Theorem 4.1. The set of respectable relations is the set of all mixed consequence relations and their intersections (including the order-theoretic relation).

4.1 Mixed consequence relations and their intersections are respectable

Three classical schemes can be used to construct consequence relations: relations of pure consequence, mixed consequence and order-theoretic consequence. Recall from section 2 that these schemes were all based on mixed consequence relations and on the intersection of some of them (see in particular Theorem 2.16). The following two facts show that all such consequence relations, intersections of mixed consequence relations, satisfy our requirements, i.e. they are respectable:

Theorem 4.2. A mixed consequence relation is respectable.

Theorem 4.3. The intersection of (possibly infinitely many) respectable consequence relations is respectable.

[7]We explore the structural character of the notion of respectability in a follow-up to this paper, in which we present a different perspective on the resulting set of consequence relations.

Proof of Theorem 4.2. Consider some mixed consequence relation $\models_{\mathcal{D}_p,\mathcal{D}_c}$. • Bivalence compliance follows from the fact that $\mathcal{D}_p$ and $\mathcal{D}_c$ contain 1 but not 0. • Truth-relationality is obtained from the relation defined as $\gamma \lhd_{\mathcal{D}_p,\mathcal{D}_c} c$ iff $\gamma \not\subseteq \mathcal{D}_p$ or $c \in \mathcal{D}_c$. • Value-monotonicity follows from the upsetness of $\mathcal{D}_p$ and $\mathcal{D}_c$. Let (Γ_1, C_1) and (Γ_2, C_2) be as in Definition 3.7. Assuming that $\Gamma_1 \models C_1$, we can prove that $\Gamma_2 \models C_2$: If $\Gamma_1(i) \not\subseteq \mathcal{D}_p$ then $\Gamma_2(i) \not\subseteq \mathcal{D}_p$ and if $C_1(i) \in \mathcal{D}_c$ then $C_2(i) \in \mathcal{D}_c$. • Validity coherence follows because $\models C$ is equivalent to $\forall i : C(i) \in \mathcal{D}_c$, which has to be true if $\forall\Gamma \neq \emptyset, \Gamma \models C$ (pick Γ a singleton proposition with all values in $\mathcal{D}_p$, e.g., $\Gamma = \{\top\}$). □

Proof of closure under intersection (Theorem 4.3). For all js in some index set J, let $\models_j$ be a respectable consequence relation. One can prove that all the properties that make up respectability carry over to $\models_J$, defined as the intersection of these consequence relations (i.e. by $\Gamma \models_J C$ iff for all j, $\Gamma \models_j C$). For instance, if $\lhd_j$ are relations which make each of the $\models_j$ truth-relational, $\models_J$ can be proved to be truth-relational through the function $\lhd_J$ which holds iff all the $\lhd_j$s hold. □

4.2 A respectable consequence relation is an intersection of mixed consequence relations

To complete the proof of the main Theorem 4.1 stating the identity of the respectable consequence relations with the intersection of mixed consequence relations, we now need to state and prove the inclusion from left to right:

Theorem 4.4. All respectable relations are intersections of mixed consequence relations.

Proof. Let $\models$ be a respectable consequence relation. We shall construct a set of mixed consequence relations which all are more inclusive than $\models$, but so that for each case where $\models$ fails, we can show that at least one of these mixed consequence relations fails. This way, their intersection will be identical to $\models$.

For λ a pair (Γ, C) made of constant propositions only (wherever convenient, we equate a constant proposition with its associated truth value) and such that $\Gamma \neq \emptyset$ and $\Gamma \not\models C$, we define:

$$\begin{cases} \mathcal{D}_p^\lambda = \{\alpha : \exists \alpha_0 \in \Gamma, \alpha \geq \alpha_0\} & \text{(this is in essence the smallest upset which contains } \Gamma) \\ \mathcal{D}_c^\lambda = \{\beta : \mathcal{D}_p^\lambda \models \beta\} \end{cases}$$

$\mathcal{D}_p^\lambda$ is an upset. (a) 1 is in $\mathcal{D}_p^\lambda$, because Γ is not empty and 1 is larger or equal to any value in Γ, so it is in $\mathcal{D}_p^\lambda$. (b) 0 is not in $\mathcal{D}_p^\lambda$. Otherwise, 0 would be in Γ (there is no smaller value which could be in Γ) and therefore Γ would $\models$-entail C (by value-monotonicity applied to $0 \models 0$): contradiction. (c) Assume that $\alpha_1 \in \mathcal{D}_p^\lambda$. Then there is $\alpha_0 \in \Gamma$ with $\alpha_1 \geq \alpha_0$. For $\alpha_2 \geq \alpha_1$, $\alpha_2 \geq \alpha_0$ by transitivity of the ordering relation and therefore α_2 is also in $\mathcal{D}_p^\lambda$.

$\mathcal{D}_c^\lambda$ is an upset. (a) 1 is in $\mathcal{D}_c^\lambda$, because any set of premises $\models$-entails the tautology (by value-monotonicity and $1 \models 1$). (b) Assume that 0 is in $\mathcal{D}_c^\lambda$, i.e. $\mathcal{D}_p^\lambda \models 0$. We can associate to every element α in $\mathcal{D}_p^\lambda$ an element of Γ smaller or equal to α. That is, Γ is a stronger premise set than $\mathcal{D}_p^\lambda$ and therefore, by value-monotonicity, $\Gamma \models C$, contradiction. (c) Value-monotonicity guarantees that for $\alpha_1 \geq \alpha_2$, $\alpha_1 \in \mathcal{D}_c^\lambda$ entails that $\alpha_2 \in \mathcal{D}_c^\lambda$.

We write $\models_\lambda$ for what we now know is a mixed consequence relation: $\models_{\mathcal{D}_p^\lambda,\mathcal{D}_c^\lambda}$. Let us prove two additional facts about it.

For $\lambda = (\Gamma, C), \Gamma \not\models_\lambda C$**.** Assume the contrary. Since Γ is included in the set of designated values for the premises $\mathcal{D}_p^\lambda$, then C must be included in the set of designated values for the conclusion $\mathcal{D}_c^\lambda$. But that means that $\mathcal{D}_p^\lambda \models C$. As above, we would then conclude that $\Gamma \models C$ (by value-monotonicity). Contradiction.

If $\Gamma' \models C'$ **then** $\Gamma' \models_\lambda C'$**.** Assume that $\Gamma' \models C'$ but $\Gamma' \not\models_\lambda C'$. Then $\Gamma' \subseteq \mathcal{D}_p^\lambda$ (the only situation which can block a mixed consequence relation to hold is if the set of premises is included in the set of designated values for premises). But then we conclude that $\mathcal{D}_p^\lambda$ $\models$-entails C' (because Γ', which is a subset of $\mathcal{D}_p^\lambda$, already $\models$-entails C' by assumption). Hence, C' is in $\mathcal{D}_c^\lambda$ by construction, but this is the set of designated values for conclusions for $\models_\lambda$ and so $\Gamma' \models_\lambda C'$: contradiction.

Hence, for constant propositions, we found a family of mixed consequence relations $\models_\lambda$ such that (i) for all cases where $\models$ holds they all hold and (ii) for each case in which $\models$ fails to hold at least one of them also fails to hold (these cases are indexed by the λs).[8] Their intersection therefore is exactly like $\models$ on constant propositions. By truth functionality, these respectable consequence relations are both fully determined by their status on constant propositions, Theorem 3.4, so they are identical. That is, we proved:

$$\models \quad = \quad \bigcap_{\lambda \in \Lambda} \models_\lambda \text{ , with } \Lambda = \{(\Gamma, C) | \Gamma \neq \emptyset, \Gamma \not\models C, \text{all constant propositions}\}$$

□

We have established that the set of respectable consequence relations is the set of mixed consequence relations and their intersections. The latter initially appeared as forming a natural class, based on an inspection of the situation in three-valued logic. Our result gives a more principled justification of that intuition, by relating mixed consequences and their intersections to a small set of fundamental properties.

5 Second characterization and enumeration in the case of well-ordered vs. degenerate truth values

In this section, we go through a second characterization of respectable consequence relations. This characterization relies on the correspondence between consequence relations and relations between truth values. It will allow us to enumerate respectable consequence relations. We will first establish general results (section 5.1) and then apply these results to logics of two extreme types: logics with a well-ordered set of truth values (section 5.2) or with a degenerate order (section 5.3).

5.1 General results

Thanks to truth-relationality, respectable consequence relations can be defined in terms of relations between sets of truth values and truth values (hence as subsets of $\mathcal{P}(\mathcal{V}) \times \mathcal{V}$). The other properties

[8]The above construction was made for non-empty sets Γ, but empty sets are not problematic by validity coherence and monotonicity of $\models$ and of all of the $\models_\lambda$ (see also Fact 3.12): $\models C$ iff $\forall \Gamma \neq \emptyset, \Gamma \models C$ iff $\forall \lambda, \forall \Gamma \neq \emptyset, \Gamma \models_\lambda C$ iff $\forall \lambda, \models_\lambda C$.

we impose on respectable consequence relations have a direct translation when stated in those terms. To see this, let us first define a pre-order on this set.

Definition 5.1. A pre-order over $\mathcal{P}(\mathcal{V}) \times \mathcal{V}$ is obtained as follows:

$$(\gamma_1, c_1) \preccurlyeq (\gamma_2, c_2) \text{ iff } \begin{cases} \forall a_1 \in \gamma_1, \exists a_2 \in \gamma_2 : a_1 \geq a_2 \\ c_1 \leq c_2 \end{cases}$$

First, it should be apparent that $\preccurlyeq$ is a pre-order. Second, it closely parallels the orders between propositions and sets of propositions given in Definition 3.6, which are the ingredients of value-monotonicity. With that pre-order in hand, we can tightly link respectable relations with upsets of $\mathcal{P}^*(\mathcal{V}) \times \mathcal{V}$, where $\mathcal{P}^*(\mathcal{V})$ represents the set of non-empty subsets of $\mathcal{V}$.[9] The following conditions are parallel to the conditions on sets of designated values in Definition 2.7.

Theorem 5.2. Respectable consequence relations are in one-to-one correspondence with subsets $\mathcal{R}$ of $\mathcal{P}^*(\mathcal{V}) \times \mathcal{V}$ such that: (i-a) $\mathcal{R}$ is not the whole set, (i-b) $\mathcal{R}$ is not empty and contains $(\{1\}, 1)$ and $(\{0\}, 0)$, (ii) $\mathcal{R}$ is an upset of $\mathcal{P}^*(\mathcal{V}) \times \mathcal{V}$, i.e. $\forall x \preccurlyeq y$, if $x \in \mathcal{R}$ then $y \in \mathcal{R}$.

Proof.

- Consider the function which associates to a respectable consequence relation $\models$ a subset of $\mathcal{P}^*(\mathcal{V}) \times \mathcal{V}$ defined as: $\mathcal{R}_{\models} = \{(\gamma, c) : \gamma \neq \emptyset \text{ and } \overline{\gamma} \models \overline{c}\}$. For a respectable $\models$, we obtain:

  (i-a) $\mathcal{R}_{\models}$ is not the whole target set because $\top \not\models \bot$ (i.e. $(\{1\}, 0) \notin \mathcal{R}_{\models}$) by bivalence-compliance of $\models$.

  (i-b) $\mathcal{R}_{\models}$ contains $(\{1\}, 1)$ and $(\{0\}, 0)$ by bivalence-compliance of $\models$.

  (ii) follows from the value-monotonicity of $\models$.

- Conversely, consider the function which associates to a subset $\mathcal{R}$ of $\mathcal{P}^*(\mathcal{V}) \times \mathcal{V}$ the consequence relation defined as: $\Gamma \models_{\mathcal{R}} C$ iff $\forall i, (\Gamma(i), C(i)) \in \widetilde{\mathcal{R}}$, with $\widetilde{\mathcal{R}} = \mathcal{R} \cup \{(\emptyset, c) | \forall \gamma \neq \emptyset, (\gamma, c) \in \mathcal{R}\}$. If $\mathcal{R}$ further satisfies the conditions (ia), (ib) and (ii) from the theorem, we obtain that $\models_{\mathcal{R}}$ satisfies the rest of the conditions of respectability:

  Truth-relationality is obtained by construction, through $\tilde{\mathcal{R}}$.

  Bivalence-compliance We must show that all elements of $\mathcal{P}^*(\{0, 1\}) \times \{0, 1\}$ behave as they should with respect to $\widetilde{\mathcal{R}}$. Most cases are obvious except for $(\{1\}, 0)$. Note that $\forall \gamma \neq \emptyset\ \forall c$, $(\{1\}, 0) \preccurlyeq (\gamma, c)$, and it follows that $(\{1\}, 0) \notin \mathcal{R}$ for otherwise by (ii) $\mathcal{R}$ would be the whole set $\mathcal{P}^*(\mathcal{V}) \times \mathcal{V}$.

  Value-monotonicity Assume that $\Gamma_1 \models_{\mathcal{R}} C_1$. Pick Γ_2, C_2 as in the Definition 3.7 of value-monotonicity. This entails that $\forall i, (\Gamma_1(i), C_1(i)) \preccurlyeq (\Gamma_2(i), C_2(i))$. We must review two cases:
  (1) $\Gamma_1 \neq \emptyset$. Then $\forall i, (\Gamma_1(i), C_1(i)) \in \mathcal{R}$. Hence, $\forall i, (\Gamma_2(i), C_2(i)) \in \mathcal{R}$ (by (ii)). And therefore $\Gamma_2 \models_{\mathcal{R}} C_2$.
  (2) $\Gamma_1 = \emptyset$. Then $\forall i, (\emptyset, C_1(i)) \in \widetilde{\mathcal{R}}$ and so $\forall i \forall \gamma \neq \emptyset, (\gamma, C_1(i)) \in \mathcal{R}$ from which it follows that $\forall i \forall \gamma \neq \emptyset, (\gamma, C_2(i)) \in \mathcal{R}$ (by (ii)). If $\Gamma_2 \neq \emptyset$, it follows that $\forall i, (\Gamma_2(i), C_2(i)) \in \mathcal{R}$ and therefore that $\Gamma_2 \models_{\mathcal{R}} C_2$. If $\Gamma_2 = \emptyset$, it follows that $\forall i, (\Gamma_2(i), C_2(i)) = (\emptyset, C_2(i)) \in \tilde{\mathcal{R}}$ and, again, that $\Gamma_2 \models_{\mathcal{R}} C_2$.

[9] The empty set can be dropped because of validity-coherence and value-monotonicity, see Fact 3.12.

Validity-coherence If $\forall \Gamma \neq \emptyset, \Gamma \models_{\mathcal{R}} C$, then $\models_{\mathcal{R}} C$ because of the extension of $\mathcal{R}$ to $\tilde{\mathcal{R}}$.

- To obtain the one-to-one correspondence, we show that the two functions above are inverse of each other. The computations are as follows:

  (1) $\Gamma \models_{\mathcal{R}_{\models}} C$
  iff $\forall i, (\Gamma(i), C(i)) \in \mathcal{R}_{\models} \cup \{(\emptyset, c) | \forall \gamma \neq \emptyset, (\gamma, c) \in \mathcal{R}_{\models}\}$
  iff $\forall i, (\Gamma(i), C(i)) \in \{(\gamma, c) : \gamma \neq \emptyset \text{ and } \overline{\gamma} \models \overline{c}\} \cup \{(\emptyset, c) | \forall \gamma \neq \emptyset, (\gamma, c) \in \{(\gamma, c) : \gamma \neq \emptyset \text{ and } \overline{\gamma} \models \overline{c}\}\}$
  iff ($\Gamma \neq \emptyset$ and $\Gamma \models C$) or ($\Gamma = \emptyset$ and $\forall i, C(i) \in \{c | \forall \gamma \neq \emptyset, (\gamma, c) \in \{(\gamma, c) : \overline{\gamma} \models \overline{c}\}\}$) (by truth-relationality)
  iff ($\Gamma \neq \emptyset$ and $\Gamma \models C$) or ($\Gamma = \emptyset$ and $\forall i, C(i) \in \{c | \forall \gamma \neq \emptyset, \overline{\gamma} \models \overline{c}\}$)
  iff ($\Gamma \neq \emptyset$ and $\Gamma \models C$) or ($\Gamma = \emptyset$ and $\models C$) (by validity-coherence and truth-relationality)
  iff $\Gamma \models C$.

  (2) $(\gamma_0, c_0) \in \mathcal{R}_{\models_{\mathcal{R}}}$
  iff $(\gamma_0, c_0) \in \{(\gamma, c) : \gamma \neq \emptyset \text{ and } \overline{\gamma} \models_{\mathcal{R}} \overline{c}\}$
  iff $\gamma_0 \neq \emptyset$ and $\overline{\gamma_0} \models_{\mathcal{R}} \overline{c_0}$
  iff $\gamma_0 \neq \emptyset$ and $\forall i, (\gamma_0(i), c_0(i)) \in \mathcal{R} \cup \{(\emptyset, c) | \forall \gamma \neq \emptyset, (\gamma, c) \in \mathcal{R}\}$
  iff $\gamma_0 \neq \emptyset$ and $(\gamma_0, c_0) \in \mathcal{R} \cup \{(\emptyset, c) | \forall \gamma \neq \emptyset, (\gamma, c) \in \mathcal{R}\}$
  iff $\gamma_0 \neq \emptyset$ and $(\gamma_0, c_0) \in \mathcal{R}$
  iff $(\gamma_0, c_0) \in \mathcal{R}$. □

As it turns out, it is possible to restrict attention to a smaller set by factoring in the particular roles of 0 (in premises) and 1 (in conclusions). In short, the fact that the tautology is a validity and that the contradiction entails any proposition makes it unnecessary to further specify the role of 0 and 1 in premises or conclusions, respectively. More precisely, we obtain the following one-to-one correspondence:

Corollary 5.3. Respectable consequence relations are in one-to-one correspondence with subsets $\mathcal{R}$ of $\mathcal{P}^*(\mathcal{V} - \{0\}) \times (\mathcal{V} - \{1\})$ such that: (i) $\mathcal{R}$ is not the whole set (i.e. $(\{1\}, 0) \notin \mathcal{R}$) and (ii) $\mathcal{R}$ is an upset, i.e. $\forall x \preccurlyeq y$, if $x \in \mathcal{R}$ then $y \in \mathcal{R}$.

Proof. An element (γ, c) of $\mathcal{P}^*(\mathcal{V}) \times (\mathcal{V})$ which is not in $\mathcal{P}^*(\mathcal{V} - \{0\}) \times (\mathcal{V} - \{1\})$ is such that either $0 \in \gamma$, and then $(\{0\}, 0) \preccurlyeq (\gamma, c)$, or $\gamma \neq \emptyset$ and $c = 1$, in which case $(\{1\}, 1) \preccurlyeq (\gamma, c)$. In both cases, (γ, c) ought to belong to the relation and it is not necessary to specify it. □

We can state another one-to-one correspondence result, which is obtained similarly by replacing the way some of the information is determined:

Corollary 5.4. Respectable consequence relations are in one-to-one correspondence with subsets $\mathcal{R}$ of $\mathcal{P}(\mathcal{V} - \{0, 1\}) \times (\mathcal{V} - \{1\})$ such that: (i) $\mathcal{R}$ is not the whole set (i.e. $(\emptyset, 0) \notin \mathcal{R}$) and (ii) $\mathcal{R}$ is an upset, i.e. $\forall x \preccurlyeq y$, if $x \in \mathcal{R}$ then $y \in \mathcal{R}$.

Proof. Here we replace the possibility of 1 being in what corresponds to the premise set by the possibility for this set to be empty. The reason why this is possible is in essence that for a value-monotonic consequence relation $\models$, for any proposition C and any (empty or non-empty) set of propositions Γ, $\Gamma \models C$ iff $\Gamma, \top \models C$. □

We have offered a new perspective on respectable relations, via the relations they correspond to at the level of truth values, rather than in terms of propositions. This approach will allow us to investigate in more detail the two extreme cases of N-valued logics, those with well-ordered truth values (section 5.2) and those with truth values based on a degenerate order (section 5.3). We give an enumeration of the set of possible consequence relations in these two extreme cases.

5.2 Logics based on a well-ordered set of truth values

We argued that value-monotonicity, which constrains the relation between an ordering on truth values and a consequence relation, is in a sense a minimal requirement since the ordering on truth values can be degenerate. Here, however, we examine more closely what happens when the set of truth values is assumed to be well-ordered (without loss of generality, we can restrict attention to a total order over a finite set of values). Note that this is not such a strong requirement either. In three-valued logic, for instance, the order will necessarily be total in virtue of the requirement that 1 be larger than anything else and that 0 be smaller than anything else.

5.2.1 Reducibility to single premises and RWO tables

A nontrivial implication of the assumption that the truth values are well-ordered is that no relation holds between a set of constant premises and a conclusion if the relation does not already hold between only one of these premises and the conclusion. It will follow that respectable consequence relations are fully determined by the *single* premise cases:

Theorem 5.5 (Reducibility)**.** In logics with a well-ordered set of truth values, consider a respectable consequence relation $\models$ and let Γ be a non empty set of constant propositions and C be a constant proposition. Then $\Gamma \models C$ iff there is $A \in \Gamma$ such that $A \models C$.

Proof. The right-to-left direction follows from value-monotonicity. For the converse direction, assume $\Gamma \models C$. Let a be the infimum of the values taken by the constant propositions in Γ. Then $\overline{a} \in \Gamma$ (the set of truth values is well-ordered) and $\overline{a} \models C$, again by value-monotonicity. $\square$

Going back to Theorem 5.2 (or its Corollary 5.3), the previous result shows that with a well-order there is no need to consider relations in $\mathcal{P}^*(\mathcal{V}) \times \mathcal{V}$ between sets of truth values and truth values. Instead, relations in $\mathcal{V} \times \mathcal{V}$ between (singletons of) truth values and truth values are sufficient. As a result, respectable relations can now conveniently be thought of in 2D. More precisely, respectable relations correspond to tables of the following form:

Definition 5.6 (Respectable Well-Ordered tables)**.** An RWO table is a relation $\mathcal{T} \subset \mathcal{V} \times \mathcal{V}$ such that:
(i-a) $\neg\mathcal{T}(1, 0)$
(i-b) $\mathcal{T}(1, 1)$ and $\mathcal{T}(0, 0)$
(ii-a) for all $a_1 \geq a_2$ and for all c, if $\mathcal{T}(a_1, c)$ then $\mathcal{T}(a_2, c)$.
(ii-b) for all $c_1 \leq c_2$ and for all a, if $\mathcal{T}(a, c_1)$ then $\mathcal{T}(a, c_2)$.

Condition (i-b) together with (ii-a) and (ii-b) can be written as $\forall x, \mathcal{T}(x, 1)$ and $\mathcal{T}(0, x)$, which makes it clear that we can restrict attention to $\mathcal{T} \subseteq (\mathcal{V} - \{0\}) \times (\mathcal{V} - \{1\})$. This corresponds to the difference between Theorem 5.2 and its Corollary 5.3. In N-valued logic, the relevant type of relation can be represented using tables of the following form, with the first argument of the relation in lines and the second in column, such that Ys (for 'Yes', i.e. inclusion in the relation, as opposed to N for 'No') spread downward (because of (ii-a)) and leftward (because of (ii-b)), the gray part corresponds to the possible reduction to $(\mathcal{V} - \{0\}) \times (\mathcal{V} - \{1\})$:

	1		...		0
1	Y				N
	Y				
$\vdots$	Y				
	Y				
0	Y	Y	Y	Y	Y

$Y \downarrow$

$\leftarrow Y$

We can thus state a visual version of our correspondence result:

Corollary 5.7. When the set of truth values is well-ordered, respectable relations are in one-to-one correspondence with RWO tables.

Proof. Informally, this result follows because: • the constraints in (i) encode bivalence-compliance, • the constraints in (ii) encode value-monotonicity and translate into the spreading of Ys downwards and leftwards, • the very fact that we restrict attention to a table with truth values (rather than propositions) encodes truth functionality. • Validity-coherence ensures that relations involving an empty set of premises are taken into account (see also Fact 3.12). □

5.2.2 The number of totally ordered N-valued respectable relations

When the number of truth values is finite and totally ordered, the correspondence of respectable consequence relations with RWO tables allows us to enumerate how many respectable consequence relations there are. The following result determines how many possibilities there are to choose from for a consequence relation to be respectable.

Theorem 5.8. In N-valued logic, there are $\frac{(2\times(N-1))!}{(N-1)!^2} - 1$ different respectable relations.

Proof. An RWO table is obtained by delineating the lower left corner of Ys from the upper right corner of Ns. This corresponds to following a path in a grid with $N-1$ steps to the right and $N-1$ steps to the bottom. These paths are defined by deciding where are the bottom steps within the $2\times(N-1)$-long path (the other steps are then right steps). There are $\binom{2(N-1)}{N-1} = \frac{(2(N-1))!}{(N-1)!(N-1)!}$ such options. There is one path that needs to be excluded though, the one which goes all the way to the right first and then straight to the bottom, because this would exclude that the top right cell be an N. □

To illustrate, in three-valued logic we obtain $\frac{(2\times(3-1))!}{(3-1)!^2} - 1 = 5$ respectable relations, which correspond to ss, tt, st, ts, $ss \cap tt$. Note that with 2 possible sets of designated values, there are 2×2 possible mixed consequence relations and in principle $2^{2\times 2} - 1 = 15$ possible ways of intersecting them into respectable consequence relations (this is the number of all subsets of potential intersections, to the exclusion of the empty set). This result thereby provides a useful demonstration that no other intersection of mixed consequence relations provides a consequence relation distinct from the five above in three-valued logics.

Truth values	N	2	3	4	5	6	7
Well-Orders	$\binom{2(N-1)}{N-1} - 1$	1	5	19	69	251	923
Degenerate Orders	Thm 5.14	1	5	83	28,925	7,663,696,588	?

Table 1. Number of consequence relations as a function of the number N of truth values ($2 \leq N \leq 7$), for well-orders and for degenerate orders.

The previous result also indicates that the number of respectable relations quickly increases as a function of the number of truth values in the logic. In 4-valued logic with a well-ordered set of truth values, for example, we obtain $\frac{(2(4-1))!}{(4-1)!^2} - 1 = 19$ respectable relations. Since there are 3 possible upsets, there are 3×3 mixed consequence relations and there is 1 order-theoretic relation. That means that there are 9 more respectable relations, corresponding to intersections of mixed consequence relations which do not reduce to mere mixed consequence relations or to the order-theoretic relation (note that one can form $2^9 - 1$ intersections out of the 9 mixed consequence relations, only 10 distinct consequence relations emerge from these 511 options though). These are new consequence relations waiting to be discovered and inspected. Importantly, some of these consequence relations will not distinguish two of the four values so that they can essentially be reduced to 3-valued consequence relations. But this is already true of mixed consequence relations in which two values will be given the same role in the sets of designated values for premises and for conclusions. In fact, one can prove that 7 distinct respectable relations in 4-valued logics do not map to any respectable relation in 3-valued logics.

Table 1 provides actual values in logics with 2 to 7 well-ordered truth values (WO), integrating results from the next section, which investigates logics with a degenerate order (DO).

5.3 Logics with a degenerate partial order: four-valued logic and beyond

Logics with a well-ordered set of truth values such as three-valued logics sit at one end of the spectrum we can consider. At the other end of the spectrum, we find logics based on a minimally constrained set of truth values, one with a degenerate order such that the only relations are that 1 is higher than any other truth value and 0 is lower than any other truth value. A well-known example is provided by Belnap's four-valued logic, in which the intermediate truth values b and n between 1 and 0 are incomparable ([2, 3, 11, 27]). In this section, we investigate such cases in more detail, following the structure of the previous section: we first look at reducibility possibilities and then count respectable consequence relations involving N truth values.

5.3.1 Weak reducibility

One consequence of dropping the well-ordering is that we lose the reducibility property from before. Concretely, respectable relations are not fully determined by single premise cases anymore. Consider the following examples:

Fact 5.9. In 4-valued logics with the set of truth values $\{1, b, n, 0\}$ with no ordering of b and n, the following respectable relations are equivalent on constant single-premise relations, but not beyond:

$$\begin{cases} \models_1 & := & \models_{\{1,b\},\{1,b\}} \cap \models_{\{1,n\},\{1,n\}} \\ \models_2 & := & \models_{\{1,b,\},\{1,b\}} \cap \models_{\{1,n\},\{1,n\}} \cap \models_{\{1,b,n\},\{1,b,n\}} \end{cases}$$

Proof. For single premises, clearly $A \models_2 B$ implies $A \models_1 B$; conversely, the additional relation $\models_{\{1,b,n\},\{1,b,n\}}$ in $\models_2$ does not rule out any $\overline{a} \models_2 \overline{c}$ potential validity which would not already be ruled out by one of the first two conjuncts, i.e. by $\models_1$. However, one can check that $\overline{b}, \overline{n} \models_1 0$ but $\overline{b}, \overline{n} \not\models_2 0$. □

Consequently, 2-dimensional tables are no longer adequate to represent a respectable relation. Instead, we have a much weaker possibility of reduction, which only allows us to reduce premise sets down to an uninteresting cardinality $N-2$ in N-valued logics:

Theorem 5.10 (Weak reducibility)**.** In N-valued logics ($N > 2$) with a degenerate order over the set of truth values, consider a respectable consequence relation $\models$ and let Γ be a non empty set of constant propositions and C be a constant proposition. Then $\Gamma \models C$ iff there are $A_1, ..., A_{N-2} \in \Gamma$ such that $A_1, ..., A_{N-2} \models C$.

Proof. The right-to-left direction follows from value-monotonicity. The converse direction follows from the fact that there is not more than N constant propositions in N-many valued logics, that $\top$ as a premise can always be replaced by any other premise (by value-monotonicity) and that if $\bot$ were to be in the set of premises we could actually drop all the others (again, by value-monotonicity). Note that for $N = 3$ we recover the full version of reducibility (Lemma 5.5). □

5.3.2 The number of N-valued respectable relations

Corollary 5.3 established in the general case that respectable relations are in one-to-one correspondence with upsets strictly included in $\mathcal{P}^*(\mathcal{V} - \{0\}) \times (\mathcal{V} - \{1\})$. With a well-order, it was possible to count how many such sets exist, because we could restrict attention to $(\mathcal{V} - \{0\}) \times (\mathcal{V} - \{1\})$. With a degenerate order, this is not a possibility. Instead, we will take advantage of the fact that no two truth values different than 0 and 1 are ordered. One consequence of this is that we will be able to use the inclusion order over $\mathcal{P}(\mathcal{V} - \{0, 1\})$, instead of the pre-order we used so far:

Fact 5.11. Our comparison measure for sets of premises Γ_1 and Γ_2 is given by the statement from Definition 3.6 or equivalently the top part of Definition 5.1: $\forall a_1 \in \Gamma_1, \exists a_2 \in \Gamma_2, a_1 \leq a_2$. If the order is degenerate and neither Γ_1 nor Γ_2 contains 0 or 1, then this amounts to $\Gamma_1 \subseteq \Gamma_2$.

With the partial order of inclusion in hand, very much in the spirit of Tarskian monotonicity, we can define the notion of an antichain (immediately using the notation for inclusion):

Definition 5.12. An antichain of a partially ordered set $S_\subseteq$ is a subset α of S such that no two distinct elements are ordered: $\forall x, y \in \alpha, x \not\subset y$.

Fact 5.13. Antichains and upsets of S are in one-to-one correspondence through the function which associates to an antichain α the upset $\hat{\alpha} = \{a : \exists a_0 \in \alpha, a_0 \subseteq a\}$.

Proof. One can check (classical) that the reciprocal function is the one which associates to an upset the set of its minimal elements (note, for instance that two minimal elements of a partially ordered set can never be ordered, for otherwise one of these two elements is not minimal). □

Theorem 5.14. The number of respectable relations in an N-valued logic based on a degenerate partial order is:

$$\sum_{\alpha \in \mathcal{A} - \{\{\emptyset\}\}} |\{\beta \in \mathcal{A} : \hat{\alpha} \subseteq \hat{\beta}\}|^{N-2},$$

with $\mathcal{A}$ the set of antichains of $\mathcal{P}(V - \{0, 1\})$, In Appendix B we show how this formula applies to logics with 3, 4 or 5 truth values. These examples may help understand the formula and its proof, to which we now turn.

Proof. In Corollary 5.4, we established that respectable relations are in one-to-one correspondence with strict upsets of $\mathcal{P}(\mathcal{V} - \{0, 1\}) \times (\mathcal{V} - \{1\})$. Here, we propose to look at such subsets $\mathcal{R}$ slice by slice, where a slice for a value x corresponds to the projection of $\mathcal{R}$ onto $\mathcal{P}(\mathcal{V} - \{0, 1\}) \times \{x\}$. Putting it differently, this corresponds to looking at each truth value x one after the other, and then see what sets of premises entail this x. Repeating a result from above, let us recall that it is not necessary to specify what happens with subsets containing 0 (because this follows from the fact that $\forall\Gamma, \Gamma, 0 \models C$) nor 1 (because $\forall\Gamma, \Gamma, 1 \models C$ iff $\Gamma \models C$). For this reason we can restrict attention to $\mathcal{P}(\mathcal{V} - \{0, 1\})$ instead of $\mathcal{P}(\mathcal{V})$.

First, one can determine which of the subsets of $\mathcal{V} - \{0, 1\}$ entail the contradiction. The empty set cannot be one of these because the contradiction is not a validity (see also (i) in Corollary 5.4). Apart from that any upset could do, and these sets can be described by antichains. This corresponds to the first summation over the antichains $\mathcal{A}$: "$\sum_{a \in \mathcal{A} - \{\{\emptyset\}\}}$".

Second, one has to repeat the same process for each intermediate value x: determine which of the subsets of $\mathcal{V} - \{0, 1\}$ entail this value x. Because the order is degenerate, this can be done *independently* for each of the indeterminate values (two indeterminate values are not ordered and so value-monotonicity does not impose any dependency constraint), hence the plain $N - 2$ exponent.

But for intermediate values, not all possible antichains will work: the set of sets of premises which entail some indeterminate value c should contain the set of sets of premises which entail the contradiction (value-monotonicity is active here because even if the order is degenerate, it is still required that $0 < c$). The $\hat{\alpha} \subseteq \hat{\beta}$ restriction ensures that this is taken into account: if some set of premises entails 0 (i.e. belongs to $\hat{\alpha}$), it entails c (i.e. it belongs to $\hat{\beta}$). □

This formula is not quite a full analytic expression of the number of respectable consequence relations. We note that enumeration of antichains in powersets is a notoriously difficult problem. For instance, the cardinals of the sets $\mathcal{A}$ for n indeterminate values is known as the n^{th} Dedekind number, and there is no analytical formula for Dedekind numbers, which are only known up to $n = 8$. But this formula can still be used to complete Table 1. The counts for $N = 3, 4$ and 5 are explained in Appendix B. We verified all the cases from $N = 3$ to $N = 6$ with a Python script available at `http://semanticsarchive.net/Archive/GQzYTM4N/Chemla-Egre-Spector-LCrelations.py`. The sequence of values for degenerate orders was unknown to the Online Encyclopedia of Integer Sequences (`http://oeis.org`) and is now listed under `https://oeis.org/A271219`.

6 Conclusions

In this paper we have proposed to characterize a natural class of consequence relations that would allow us to unify the three schemes used to construct logical consequence that we started with: namely pure consequence, order-theoretic consequence and mixed consequence. Pure consequence is by definition a special case of mixed consequence, and in the three-valued case it is easy to see that order-theoretic consequence is the intersection of two pure consequence relations. But the problem for us was to find a more abstract characterization, based on more general and separate constraints. The set of all mixed consequence relations and their intersections, including the order-theoretic consequence relation, is the set of relations which naturally emerge from our constraints on what we called *respectable* relations. This set is closed under intersection, but it is not closed under union, and our result can be used to better explain the hunch one might have that the union of two well-behaved consequence relations is not automatically well-behaved.

Our method selects consequence relations in what seems to be exactly the right way in three-valued logic for instance, but some calculations revealed curious results. We enumerated all respectable relations in N-valued logic, and as such, we could count all distinct intersections of mixed consequence relations. These numbers quickly become astronomical, especially when the set of truth values is not well-ordered.

Because of that, one may wonder if the class of respectable consequence relations we identified is not too large relative to the broader agenda of characterizing logical operations proper. Some additional criteria might be considered for a consequence relation to count as logical (setting aside Tarski's conditions, which would preclude our admission of mixed consequence relations). One such constraint is what we call the constraint of *representability*, namely for a consequence relation to be associated with a binary conditional operator satisfying the deduction theorem (see [31] for similar considerations). Some of our respectable consequence relations are not representable when the associated set of truth values is not well-ordered. More generally, in this paper we investigated consequence relations without looking at the effect of introducing specific operators (such as negation, conjunction, or indeed a conditional operator), nor did we impose specific expressive limitations. However, we can expect distinct respectable consequence relations to collapse into one when expressiveness is constrained (see Appendix A for some examples).

On the other hand, despite the large number of consequence relations we obtain, our constraints may also be viewed as too restrictive: not all consequence relations that have been thought about are captured. For example, we assumed that a consequence relation should be monotonic, and one may wonder what consequence relations become available (and what not) if we allow ourselves to investigate non-monotonic logics, even as truth-relationality is retained (see section 3.6) — this is not of minor interest.

In short: we achieved our goal of finding all possible relations satisfying certain assumptions, but our hope is that the current framework can be fruitfully applied when only a subset of those assumptions remains relevant. Relatedly, it is apparent that some of our constraints, like validity-coherence or nontriviality, are purely structural, whereas others, like truth-relationality, value-monotonicity, let alone bivalence-compliance, rest on a semantic basis. Whether all of our constraints could be stated in purely structural terms is a problem we are currently investigating. We leave that matter, as well as the issue of representability, for a different paper.

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Appendix A Relativizing respectability to expressible propositions

In this appendix we clarify whether our characterization of respectable consequence relations defined over the set of all possible propositions provides us with the correct characterization of respectable consequence relations defined over the set of all *expressible* propositions of a certain interpreted language. This matters since logical consequence is usually viewed as a relation between sentences of a formal language, and the language in question need not be expressively complete. First, we need to make clear what the question is: to do so we propose a first definition of respectability applicable to proper subsets of the the set of all propositions. We show that this first attempt fails to provide appropriate generalizations. We then state a second definition which gives a way around that difficulty.

A.1 Respectability relative to a non-maximal language, first attempt

Definition A.1. Let $PROP$ be the set of functions $\mathcal{V}^{\mathcal{I}}$, with $\mathcal{I}$ a nonempty index set and $\mathcal{V}$ a truth value set as previously defined. Let $\mathcal{L}$ be a subset of $PROP$. Let us define an $\mathcal{L}$-consequence relation $\models$ as a subset of $\mathcal{P}(\mathcal{L}) \times \mathcal{L}$. We call an $\mathcal{L}$-consequence relation $\models$ $\mathcal{L}-$*respectable* if the following four conditions hold:

(i) $\mathcal{L}$-bivalence-compliance: If Γ is a set of bivalent propositions that belong to $\mathcal{L}$, and if C is a bivalent proposition that belongs to $\mathcal{L}$,

$$\Gamma \models C \quad \text{iff} \quad \Gamma \models_c C$$

(ii) $\mathcal{L}$-truth-relationality: There is a relation $\triangleleft$ between sets of truth-values and truth-values (i.e. a subset of $\mathcal{P}(\mathcal{V}) \times \mathcal{V}$) such that, for every $(\Gamma, C) \in \mathcal{P}(\mathcal{L}) \times \mathcal{L}$,

$$\Gamma \models C \quad \text{iff} \quad \forall i \in \mathcal{I}, \{P(i) | P \in \Gamma\} \triangleleft C(i)$$

(iii) $\mathcal{L}$-value-monotonicity: For every $(\Gamma, C) \in \mathcal{P}(\mathcal{L}) \times \mathcal{L}$,

$$\Gamma \models C \quad \text{iff} \quad \forall (\Gamma', C') \in \mathcal{P}(\mathcal{L}) \times \mathcal{L} \text{ such that } \Gamma' \leq \Gamma \text{ and } C \leq C' : \Gamma' \models C'.$$

(iv) $\mathcal{L}$-validity-coherence: For all propositions $C \in \mathcal{L}$, if $\forall \Gamma \in \mathcal{P}^*(\mathcal{L}), \Gamma \models C$ then $\models C$.

The general notion of respectability corresponds to the notion of $PROP$-respectability. There are various meaningful ways to extend $\mathcal{L}$-consequence-relations into plain consequence relations (see below), but there is one privileged way to restrict a consequence relation into an $\mathcal{L}$-consequence relation:

Definition A.2. Given a relation R between sets of propositions and propositions, i.e. a subset of $\mathcal{P}(PROP) \times PROP$, we note $R^{\mathcal{L}}$ its restriction to $\mathcal{L}$. That is, $R^{\mathcal{L}} := R \cap (\mathcal{P}(\mathcal{L}) \times \mathcal{L})$.

The following facts describe which properties constitutive of respectability are preserved under restriction:

Fact A.3. If $\mathcal{L}$ is a subset of $PROP$ and if $\models$ is bivalence-compliant, truth-relational, and value-monotonic, then so is $\models^{\mathcal{L}}$.

Fact A.4. A respectable consequence relation $\models$ may not remain validity-coherent when restricted to a subset $\mathcal{L}$ of $PROP$.

Proof of Fact A.3. The three conditions of bivalence-compliance, truth-relationality and value-monotonicity are all universally quantified statements over members of $\mathcal{L}$ and sets of members of $\mathcal{L}$. If these universal statements are true with respect to $PROP$, they are also true with respect to any subset of $PROP$. $\square$

Proof of Fact A.4. Let $\mathcal{L}$ be the subset consisting of the single constant proposition $\overline{\frac{1}{2}}$, with $\mathcal{V} = \{1, \frac{1}{2}, 0\}$. Clearly, $\overline{\frac{1}{2}} \models^{\mathcal{L}}_{ss} \overline{\frac{1}{2}}$, and therefore for all $\Gamma \in \mathcal{P}^*(\mathcal{L})$, $\Gamma \models^{\mathcal{L}}_{ss} \overline{\frac{1}{2}}$, but $\not\models^{\mathcal{L}}_{ss} \overline{\frac{1}{2}}$. $\square$

Respectability thus fails to be preserved by restriction to arbitrary languages, because validity-coherence may fail. This may not be seen as very problematic, since validity-coherence was introduced as a property that best applies when many propositions are expressible (cf. the comparison with Reasoning by Cases). One could thus consider that this failure is not problematic. But there is another reason why the current definition of $\mathcal{L}$-respectability is not satisfying: conversely, the following fact shows that some $\mathcal{L}$-respectable relations cannot be recovered as the restriction of any plain respectable consequence relation.

Fact A.5. For some index sets and some truth-value sets, there exists a set of propositions $\mathcal{L}$ and an $\mathcal{L}$-respectable relation $\models$ between $\mathcal{P}(\mathcal{L})$ and $\mathcal{L}$ such that there is no respectable relation $\models'$ between $\mathcal{P}(PROP)$ and $PROP$ such that $\models'^{\mathcal{L}} = \models$.

Proof. Consider an index set that consists of just three elements, i_1, i_2 and i_3, and a set of three truth-values $\{0, \frac{1}{2}, 1\}$, with their standard order. Let $\mathcal{L}$ be $\{\frac{1}{2}01, \frac{1}{2}10\}$, where the propositions are named from the values they assign to the indices i_1, i_2, i_3, in that order. Finally, consider the universal $\mathcal{L}$-consequence relation $\models$ over that language.

$\mathcal{L}$-value-monotonicity, $\mathcal{L}$-truth-relationality and $\mathcal{L}$-validity-coherence are all obvious for this universal relation; $\mathcal{L}$-bivalence-compliance holds because there is no bivalent proposition in $\mathcal{L}$.

However, there is no way of conservatively extending $\models$ into a respectable relation $\models'$ defined over the set of all possible propositions. Let us see why. Suppose there were such a relation $\models'$. Since $\models'$ is a conservative extension of $\models$, $\frac{1}{2}01 \models' \frac{1}{2}10$. By truth-relationality and Theorem 3.4, applied at index i_3, it follows that $\overline{1} \models \overline{0}$, which violates bivalence-compliance.[10] $\square$

A.2 Respectability* relative to a non-maximal language, second attempt

By examining the previous counterexample, we can see that something is in a sense missing in our definition of respectability when we apply it to a proper subset of all possible propositions.

[10] In this counter-example, $\models$ violates non-triviality, the property defined in Definition 3.17, and of which Theorem 3.18 shows that it can replace bivalence-compliance for maximally expressive languages. We can prove that even if a version of $\mathcal{L}$-non-triviality was used instead of $\mathcal{L}$-bivalence compliance, we could construct a similar counter-example. To do so, add to the language in the counter-example from the proof the proposition $\perp$, and to the relevant $\mathcal{L}$-consequence relation that $\perp$ entails all propositions and is entailed by no proposition but itself. The resulting consequence relation is no longer universal. The reader can actually prove that it satisfies $\mathcal{L}$-truth-relationality, $\mathcal{L}$-value-monotonicity, $\mathcal{L}$-validity-coherence and $\mathcal{L}$-non-triviality, but cannot be conservatively extended into a respectable consequence relation.

The crucial feature of this example is that the relation $\lhd$ which makes $\models$ truth-relational relative to $\mathcal{L}$ is itself not 'well behaved' with respect to classical truth-values, since we have $0 \lhd 1$. This does not prevent the relation $\models$ from being bivalence-compliant relative to $\mathcal{L}$, simply because $\mathcal{L}$ does not contain any bivalent proposition. A natural solution thus suggests itself: we could directly encode bivalence-compliance, validity coherence and value-monotonicity as constraints on the relation $\lhd$ which makes a consequence relation truth-relational. We thus suggest the following revised definition of $\mathcal{L}$-respectability.

Definition A.6. Let $\mathcal{I}$ be a non-empty index set and $\mathcal{V}$ be a truth-value set. Let $\mathcal{L} \subseteq \mathcal{V}^{\mathcal{I}}$.
A relation $\models$ between subsets of $\mathcal{L}$ and elements of $\mathcal{L}$ is a *$\mathcal{L}$-respectable** if there is a relation $\lhd$ between subsets of $\mathcal{V}$ and elements of $\mathcal{V}$ such that:

(i) for every $\Gamma \in \mathcal{P}(\mathcal{L})$ and every $C \in \mathcal{L}$, $\Gamma \models C$ iff $\forall i, \Gamma(i) \lhd C(i)$ [$\mathcal{L}$-truth-relationality]

(ii) For every set γ included in $\{0,1\}$ and every c in $\{0,1\}$, $\gamma \lhd c$ iff $c = 1$ or $0 \in \gamma$ [bivalence-compliance*]

(iii) if $(\gamma_1, c_1) \preccurlyeq (\gamma_2, c_2)$ and $\gamma_1 \lhd c_1$, then $\gamma_2 \lhd c_2$ (see Definition 5.1) [value-monotonicity*]

(iv) if for all (γ, c) with $\gamma \neq \emptyset$, $\gamma \lhd c$, then $\emptyset \lhd c$ (see Definition 3.10) [validity-coherence*]

This definition minimally deviates from the previous definition, and for a wide variety of languages they coincide entirely. In particular, we have that:

Fact A.7. Whenever $\mathcal{L}$ contains all constant propositions, $\mathcal{L}$-respectability and $\mathcal{L}$-respectability* coincide.

Proof. This follows from the correspondence between the $\lhd$ relation between truth-values and the $\models$ relation between corresponding constant propositions, as in Theorem 3.4. □

More importantly, we can now guarantee that $\mathcal{L}$-respectable* relations can be extended in respectable consequence relations (the counterexample we exhibited above is not $\mathcal{L}$-respectable*, it was made truth-relational by a relation that did not satisfy clause (ii) of this definition):

Fact A.8. All $\mathcal{L}$-respectable* consequence relations can be conservatively extended into a respectable consequence relation defined over the set of all possible propositions.

Proof. To construct a conservative extension of an $\mathcal{L}$-respectable* consequence relation $\models$ defined over a set of propositions $\mathcal{L}$, we simply need to pick a relation $\lhd$ that satisfies Definition A.6, and consider the truth-relational consequence relation defined on the basis of $\lhd$. By construction, this consequence relation is plainly respectable. □

We noted in Corollary 3.5 that respectable consequence relations (over all possible propositions) are such that the relation $\lhd$ which makes them truth-relational is unique. This uniqueness is not guaranteed in general for $\mathcal{L}$-respectability*, since, for instance, classical logical consequence over the set of bivalent propositions can be extended in five different ways over the set of all trivalent propositions while preserving respectability* (but it is guaranteed as soon as $\mathcal{L}$ includes all the constant propositions).

Appendix B Counting respectable consequence relations in logics with a degenerate order

The counts are made by listing which sets of sets of premises may entail the contradiction and which may entail an intermediate value. Following Theorem 5.14, we proceed in several steps: (1) we identify the set $\mathcal{A}$ of all antichains of $\mathcal{P}(\mathcal{V}-\{0,1\})$, any antichain α corresponds to a possible set of sets of premises ($\hat{\alpha}$) which may entail a given conclusion, (2) we pick one such antichain, α, different from $\{\emptyset\}$, and reason in the case where this antichain represents the sets of premises which entail the value 0, (3) for each such α, we count the number of antichains β which can be associated to an intermediate value given that α is associated to 0, i.e. which $\beta \in \mathcal{A}$ is such that $\hat{\alpha} \subseteq \hat{\beta}$ (or equivalently, $\forall a \in \alpha, \exists b \in \beta, b \subseteq a$). Visually, we represent the elements of $\mathcal{A}$ in lines for the possible αs and in columns for the possible βs. If a combination of α and β is possible we mark it as such. For each line (i.e. for each α) we obtain a number, the number of respectable LC relations is the sum of these numbers raised to the power $N-2$.

B.1 Three-valued logics

In the case of three-valued logic, we obtain our usual 5 cases. Any order here has to be the total order (because the indeterminate is necessarily ordered with respect to the other truth values, which are only 0 and 1). The formula from Corollary 5.14 applies as $5 = 3^{(3-2)} + 2^{(3-2)}$. In the table, we write *nothing* for the empty antichain, representing the fact that no subset of $\mathcal{P}(\mathcal{V}-\{0,1\})$ entails the relevant conclusion. The antichain $\{\emptyset\}$ on the other hand represents the fact that the relevant conclusion is a validity (entailed by an empty set of premises).

	nothing	$\{\#\}$	$\{\emptyset\}$	Counts
nothing	ts	$ss \cap tt$	tt	3
$\{\#\}$		ss	st	2
$\{\emptyset\}$	*excluded*			

B.2 Four-valued logics

The formula from Corollary 5.14 yields $6^2 + 5^2 + 3^2 + 3^2 + 2^2 = 83$ respectable relations.

	nothing	$\{n,b\}$	$\{n\}$	$\{b\}$	$\{n\}$, $\{b\}$	$\{\emptyset\}$	Counts
nothing	✓	✓	✓	✓	✓	✓	6
$\{n,b\}$		✓	✓	✓	✓	✓	5
$\{n\}$			✓		✓	✓	3
$\{b\}$				✓	✓	✓	3
$\{n\}$, $\{b\}$					✓	✓	2
$\{\emptyset\}$	*excluded*						

B.3 Five-valued logic

The formula from result 5.14 yields $28925 = 20^3+19^3+3\times14^3+3\times11^3+9^3+3\times6^3+3\times5^3+3\times3^3+2^3$. Below we use a, b and c for the name of the indeterminate values (recall that they are not ordered, if not larger than 0 and lower than 1).

	nothing	$\{a,b,c\}$	$\{b,c\}$	$\{a,c\}$	$\{a,b\}$	$\{a,b\}$, $\{b,c\}$	$\{a,c\}$, $\{b,c\}$	$\{a,b\}$, $\{a,c\}$	$\{a,b\}$, $\{a,c\}$, $\{b,c\}$	$\{c\}$	$\{a\}$	$\{b\}$	$\{c\}$, $\{a,b\}$	$\{a\}$, $\{b,c\}$	$\{b\}$, $\{a,c\}$	$\{b\}$, $\{c\}$	$\{a\}$, $\{c\}$	$\{a\}$, $\{b\}$	$\{a\}$, $\{b\}$, $\{c\}$	$\{\emptyset\}$	
nothing	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	20
$\{a,b,c\}$		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	19
$\{b,c\}$			✓			✓	✓		✓	✓		✓	✓	✓	✓	✓	✓	✓	✓	✓	14
$\{a,c\}$				✓			✓	✓	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓	14
$\{a,b\}$					✓	✓		✓	✓		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	14
$\{a,b\}$, $\{b,c\}$						✓			✓			✓	✓	✓	✓	✓	✓	✓	✓	✓	11
$\{a,c\}$, $\{b,c\}$							✓		✓	✓			✓	✓	✓	✓	✓	✓	✓	✓	11
$\{a,b\}$, $\{a,c\}$								✓	✓		✓		✓	✓	✓	✓	✓	✓	✓	✓	11
$\{a,b\}$, $\{a,c\}$, $\{b,c\}$									✓				✓	✓	✓	✓	✓	✓	✓	✓	9
$\{c\}$										✓			✓			✓	✓		✓	✓	6
$\{a\}$											✓			✓			✓	✓	✓	✓	6
$\{b\}$												✓			✓	✓		✓	✓	✓	6
$\{c\}$, $\{a,b\}$													✓			✓	✓		✓	✓	5
$\{a\}$, $\{b,c\}$														✓			✓	✓	✓	✓	5
$\{b\}$, $\{a,c\}$															✓	✓		✓	✓	✓	5
$\{b\}$, $\{c\}$																✓			✓	✓	3
$\{a\}$, $\{c\}$																	✓		✓	✓	3
$\{a\}$, $\{b\}$																		✓	✓	✓	3
$\{a\}$, $\{b\}$, $\{c\}$																			✓	✓	2
$\{\emptyset\}$	*excluded*																				