Identity and Quantification

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Abstract

It is a philosophical commonplace that quantification involves, invokes, or presupposes, the relation of identity. There seem to be two major sources for this belief: (1) the conviction that identity is implicated in the phenomenon of bound variable recurrence within the scope of a quantifier; (2) memories of Quine's insistence that quantification requires absolute identity for the values of variables. With respect to (1), I show that the only extant argument for a dependence of variable recurrence on identity, due to John Hawthorne, fails. I further show that the function of variable recurrence is not subsumed under that of identity, so that a dependence of the former on the latter, if any, would have to be of a rather indirect nature. With respect to (2), I argue that the relevant passage in Quine fails to establish a connection between quantification and the identity relation, and indeed wasn't intended by Quine to do so.

Much has been made of an alleged intimate link between quantification and the relation of identity, that is, "the relation that each thing has to itself and to nothing else" (Hawthorne 2003: 99). Thus, according to Colin McGinn (2000: 11), "the apparatus of variable-binding, or pronominal anaphora, invokes the notion of identity;" and according to John Hawthorne (2003: 100), "[w]ithout mastery of the concept of identity it is not clear how we would understand the significance of the recurrence of a variable within the scope of a quantifier."

Hawthorne buttresses his claim by an appeal to Quine (1964). Immediately following the sentence just quoted, linking variable recurrence to the relation of identity, he writes (Hawthorne 2003: 100, ellipsis and reference to Quine in the original): "In this vein, Quine observes that 'Quantification depends on there being values of variables, same or different absolutely. . . ' (Quine 1964: 101)". Along similar lines, Harold Noonan and Ben Curtis (2014) contend: "The basicness of the notion of identity in our conceptual scheme, and, in particular, the link between identity and quantification has been particularly noted by Quine (1964)."

I will argue, first, that neither McGinn's nor Hawthorne's reasoning succeeds in establishing a link between bound variable recurrence and identity; second, that any such link, if it exists, would have to be of a highly indirect nature, since variable recurrence and identity are expressively independent notions; and third, that the relevant passage in Quine (1964) does not support the claim that quantification presupposes identity, and indeed wasn't intended by Quine to support it.

By thus clarifying the relationship between quantification and identity—both fundamental notions of logic, metaphysics, and semantics—this paper aims to contribute to a better understanding of the basic fabric of our conceptual scheme. Insofar as it defends the independence of quantification from identity, it can also be regarded as a prolegomenon to my (2012), which aims to demonstrate that the identity relation is dispensable for all logical, semantical, and metaphysical purposes, a project that would be doomed from the start if the absence of a binary identity relation rendered quantification over objects impossible.

1 McGinn and Hawthorne

McGinn backs up his claim that variable-binding involves identity by noting that "if we say 'for some *x*, *x* is *F* and *x* is *G*' we are making tacit appeal to the idea of identity in using '*x*' twice here: it has to be the *same* object that is both *F* and *G* for this formula to come out true" (2000: 8; emphasis in the original).

Presumably the sentence following the colon is supposed to provide evidence for the involvement of identity in the truth conditions of the cited formula, and the emphasis on "same" suggests that McGinn takes this word to make the appeal to identity explicit. This might seem initially plausible: Isn't it natural to take, say,

Rick and Victor are in love with the same woman

as being identical in meaning with

The woman Rick loves is identical with the woman Victor loves?

Natural it may be; correct it is not.[1] For suppose Rick loves Ilsa and Yvonne, and Victor loves Ilsa and Zelda. Someone who utters the first sentence

[1] This was first pointed out, I believe, by Humberstone and Townsend (1994: 255), who provide examples similar to those used below.

under such circumstances is saying something true; but an utterance of the second sentence, under the same circumstances, is false or at least infelicitous, for failure of uniqueness, since there is no unique woman whom Rick loves, and likewise no unique woman Victor loves.

The example suggests another, less obvious way in which identity might be invoked by "same" talk: The definite article in "the same woman" would seem somehow to indicate uniqueness, for the expression of which we need identity, at least if we work in some standard version of first-order logic. As we've seen, it cannot be the uniqueness of Rick's beloveds; nor can it be the uniqueness of Victor's. But perhaps it is the uniqueness of beloveds common to Rick and Victor?

Alas, that won't do either. For suppose now that Rick loves Ilsa, Yvonne, and Zelda, while Victor loves Ilsa, Xenia, and Zelda. If we had to make a list of pairs of men who are in love with the same woman, Rick and Victor would no doubt be on it; in other words, it would still be correct to say that Rick and Victor are in love with the same woman, even though there isn't just one woman they both love. It seems, then, that the definite article in "the same" does not, at least in the kind of construction we've been considering, indicate uniqueness of any kind.

If this is so, McGinn has given us no reason to think that the recurrence of the variable in "for some x, x is F and x is G" has anything to do with the identity relation, for saying that *the same* x is both F and G apparently no more invokes identity than does saying that *some* x is both F and G.

Hawthorne provides a more promising argument along similar lines. He reasons as follows.

> Consider, for example the following two simple sentences of first-order predicate logic:
>
> $$\exists x \exists y\, (Fx \text{ and } Gy)$$
>
> $$\exists x\, (Fx \text{ and } Gx).$$
>
> Both require that there be at least one thing in the domain of the existential quantifier that is *F* and that there be at least one thing in the domain of the existential quantifier that is *G*. But the second sentence makes an additional requirement: that one of the things in the domain that is *F* be identical to one of the things in the domain that is *G*. Without mastery of the concept of identity it is not clear how we would understand the significance of the recurrence of a variable within the scope of a quantifier. (Hawthorne 2003: 99–100)

In order to facilitate discussion of this passage, let us note that four first-order sentences appear to play a role:

(1) $\exists x \exists y\, (Fx \wedge Gy)$

(2) $\exists x\, (Fx \wedge Gx)$

(3) $\exists x\, Fx \wedge \exists y\, Gy$

(4) $\exists x \exists y\, (Fx \wedge Gy \wedge x = y)$

It seems fair to paraphrase Hawthorne's argument as follows.

(a) Both (1) and (2) imply (3) in first-order logic without identity, hence *a fortiori* in first-order logic with identity.

(b) Sentence (2) implies (4) in first-order logic with identity, but sentence (1) does not.

(c) The concept of identity is needed to understand (4), because the identity sign explicitly occurs in (4).

(d) Understanding (2), and specifically, understanding the significance of the repeated use of the variable x in (2), requires understanding (4).

(e) Therefore, understanding (2) requires the concept of identity.

Claims (a) and (b) are unobjectionable, though (a) isn't actually relevant to the argument. Slightly stronger statements than (a) and (b) hold: (1) doesn't just imply, but is logically equivalent to (3), whether in first-order logic with or without identity; and (2) doesn't just imply, but is logically equivalent to (4) in first-order logic with identity.

It may be worth noting that Hawthorne is wise not to rest content with (b). As Humberstone and Townsend (1994: 246) observe, it won't do to claim that (2) involves identity simply because it entails a sentence containing the identity sign—after all, *every* first-order sentence implies, indeed is logically equivalent to, one in which the identity sign occurs. In particular, (1) is logically equivalent to, e.g., $\exists x \exists y \exists z\,(Fz \wedge Gy \wedge x = z)$ in first-order logic with identity. So implying a sentence in which the identity sign occurs, or even being logically equivalent to such a sentence, isn't a criterion on which (1) and (2) differ.

Premise (c) deserves extended discussion, but for present purposes, we will simply accept it, if only for the sake of argument.

Where Hawthorne's argument runs into trouble is in part (d). The problem is that the inclusion of the identity sign in (4) does nothing to help us understand the significance of x's recurrence in (2), since that very variable still recurs in (4). Why this is problematic can be brought out graphically by noting that Hawthorne's argument leads to a vicious regress: If we can only understand

(2) $\exists x\, (F\underline{x} \text{ and } G\underline{x})$,

with its recurrent variable, by means of

(4) $\exists x \exists y\, (F\underline{x} \text{ and } Gy \text{ and } \underline{x} = y)$,

then presumably we can only understand (4), in which the variable "x", after all, still recurs, by means of

(5) $\exists x \exists y \exists z (F\underline{x} \text{ and } Gy \text{ and } z = y \text{ and } \underline{x} = z)$.

But of course "x" *still* recurs in (5), and that feature won't go away no matter how many times we iterate the algorithm. We're off on an infinite regress—and we haven't even begun to address the recurrence of "y" in (4) and (5), or, for that matter, the recurrence of "z" in (5). So if we follow Hawthorne, there are infinitely many sentences we need to understand in order to understand (2), which is absurd. What makes the regress possible is of course precisely the fact that the original paraphrase of (2) as (4) just doesn't eliminate the explanandum, that is, the recurrence of the bound variable.[2]

[2]Alexis Burgess (forthcoming) points out that it is also unclear why we should feel compelled to cash out the semantic difference between (1) and (2) in terms of their impli-

Thus neither McGinn nor Hawthorne have given us a compelling, or even plausible, argument that would connect the phenomenon of bound variable recurrence to the relation of identity.

2 Variable Recurrence and Identity

That McGinn and Hawthorne have failed to identify a connection between variable recurrence and identity does not by itself mean that there isn't one. In this section I will show that variable recurrence and identity are expressively independent features of first-order logic, with neither able to fulfill the role of the other. This would seem substantially to complicate the case for an argument from variable recurrence to identity.

Let us begin by asking why the phenomenon of variable recurrence has seemed to some so suggestive of the identity relation. Presumably the thought is that we must be able to recognize, in (2), the variable "x" as the same in the argument places of F and G, respectively. So there is a notion of sameness involved, and we also have *two* items—the two occurrences of "x"—that are somehow "same-related". It thus appears that we are dealing with a binary relation of sameness, and the conjecture that this must be identity suggests itself. Let's call this the Tempting Idea.

The reasoning behind the Tempting Idea, however, doesn't hold up to scrutiny. For the items that are supposedly related, the two *occurrences* of "x", clearly aren't numerically identical; otherwise they wouldn't be *two*

cational relationships to (4). As Burgess notes, (1), but not (2), follows from the premises Fa and Gb. This seems a perfectly natural explanation of the semantic difference between (1) and (2), in which the identity relation, however, does not figure at all.

occurrences, but just one. The only way to make sense of talk of sameness here is again by means of a statement that has the form of (2): There is a symbol, "x", an occurrence of which follows F, and an occurrence of which follows G.

It is, of course, irrelevant here that the symbol "x", as well as each of its occurrences in (2), must be self-identical, for this observation hardly distinguishes (2) from (1), and in any case, it doesn't establish that the binary *relation* of identity, as opposed to the unary *property* of self-identity, figures into the situation at all.

But perhaps the most telling objection to the line of argument we've been considering consists in pointing out that the recurrence of a variable in (2) is a mere accident of the particular notation we have chosen for the languages of first-order logic. If, for instance, we cast first-order logic in the form of Quine's (1960) language of predicate functor logic (PFL), the recurrence-exhibiting sentence (2) becomes the PFL-sentence

(2p) $\mathsf{Der}\,\mathsf{Ref}\,(F \times G)$,

where not a single symbol occurs twice.

The details of PFL notation are recalled in the appendix, but roughly speaking, we can explain (2p) relative to standard notation as follows: The operation $\times$ produces, from F and G, the binary predicate $(Fy \wedge Gz)$, the functor Ref then merges the two variables into one in order to give us $(Fx \wedge Gx)$, and the functor Der existentially quantifies the remaining variable. The role of variable repetition in (2), we may say, has been made explicit in (2p) by dint of the reflection functor Ref.

One might try to argue that the variable-recurrence feature isn't really absent from (2p), or from PFL notation more generally, on the basis that the description of the interpretation of Ref, in the metalanguage, requires variable recurrence. To see the point, note that the interpretation **Ref** of the reflection functor Ref is the operator that maps any characteristic function g of a set of $(n+1)$-tuples to the characteristic function of a set of n-tuples, as follows: **Ref** g maps an n-tuple $(a_1, \ldots, a_n)$ to the value $g(a_1, \ldots, a_n, a_n)$. In the metalanguage, that is, we're making use of variable repetition.

But this is only the result of our using a metalanguage in standard notation. Since standard notation and PFL are expressively equivalent, we could in principle work with PFL in the metalanguage as well, in which case we'd simply describe the working of object-linguistic Ref in terms of metalinguistic **Ref**: The interpretation of Ref is **Ref**, period. No variable recurrence here.[3]

Note, too, that the Tempting Idea cannot be rescued by insisting that Ref itself (or recurrent variables in standard notation, for that matter) stands for the identity relation. After all, Ref is a functor, not a predicate, and as such represents an operation *on* relations, not a relation. Indeed, it is the role of PFL's identity predicate I, not of Ref, to stand for the identity relation, much as it is the role of the identity predicate "=", and not of variable recurrence, to stand for the identity relation in standard notation.[4]

[3]To expand a bit: PFL has a finite lexicon and a compositional semantics, as can be gleaned from Appendix B. There is thus no reason why PFL should not be a learnable language, and anyone brought up to learn to speak PFL would naturally formulate PFL's metalanguage in, well, PFL.

[4]See John Burgess (2005: 53–54) for a suggestion on how to pronounce PFL-predicates; in particular, note that the Ref functor plays the role of the prefix *self* that turns a two-place

So we might as well put the Tempting Idea to rest: That the variable "x" occurs twice within the scope of the quantifier in (2) is not by itself indicative of the involvement of a binary relation of identity.

Now the failure of the Tempting Idea still doesn't mean that variable recurrence within the scope of a quantifier cannot invoke the identity relation. We do know, of course, that variable recurrence is unable to perform *all* the functions of the identity relation: First-order logic without identity, which freely admits variable recurrence, is strictly less expressive than first-order logic *with* identity; for instance, finite numerical quantification is expressible in the latter but not in the former.[5]

This leaves open the possibility that the function of variable recurrence could be performed by the identity relation, in other words, that variable recurrence represents only a *partial* deployment of the identity relation.[6] But this, too, is not the case: First-order logic with identity but without variable recurrence within the scope of a quantifier is strictly less expressive than full first-order logic with identity.[7] A proof of this fact is provided in Appendix A.

At this point it is very difficult to see how claims to the effect that

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predicate such as *destroys* into the one-place predicate *self-destructs*, rather than the role of an *is* of identity.

[5]Indeed, the collection of all finite numerical quantifiers, say of the form *there are at least n objects such that*, is equivalent, over first-order logic without identity, to identity. See Wehmeier (2008: 368) for a proof.

[6]I am indebted to Lloyd Humberstone (p.c.) for suggesting this line of argument.

[7]In terms of predicate functor logic, this means that PFL without the functor Ref but with the identity predicate I is strictly less expressive than full PFL.

variable recurrence presupposes identity might be upheld: Variable recurrence is neither equivalent to identity, nor is it subsumed under identity, or even the combination of quantification and identity. What else could a dependence of variable recurrence on identity possibly consist in?

3 The Appeal to Quine

I will now show that references to Quine (1964) are misleading when they purport to bolster the claim that quantification presupposes identity. To begin with, let us take note of the context in which Hawthorne appeals to Quine—we've seen most of this passage already.

> Why is the concept of identity so basic? The point is not that we have inevitable need for an 'is' of identity in our language. Our need for the concept of identity far outstrips our need to make explicit claims of identity and difference. Consider, for example the following two simple sentences of first-order predicate logic:
>
> $$\exists x \exists y\, (Fx \text{ and } Gy)$$
>
> $$\exists x\, (Fx \text{ and } Gx).$$
>
> Both require that there be at least one thing in the domain of the existential quantifier that is *F* and that there be at least one thing in the domain of the existential quantifier that is *G*. But the second sentence makes an additional requirement: that one of the things in the domain that is *F* be identical to one of the things in the domain that is *G*. Without mastery of the

> concept of identity it is not clear how we would understand the significance of the recurrence of a variable within the scope of a quantifier. In this vein, Quine observes that 'Quantification depends on there being values of variables, same or different absolutely...' (Quine 1964: 101). [Ellipsis and parenthetical reference in the original.]

Hawthorne's use of the phrase "in this vein" suggests that the argument he has just propounded is related to Quine's point in the cited reference. As we will see, this is not at all the case. Not only does Quine's *reasoning* have nothing to do with Hawthorne's; it doesn't even purport to support the *claim* made by Hawthorne. But let's examine the text itself.

Quine (1964) is a review of Geach's (1962) *Reference and Generality*. The passage from which Hawthorne quotes is concerned with Quine's objections to Geach's doctrine of relative identity, specifically the contention that objects a and b may be identical relative to some concept F but fail to be identical relative to another concept G. It reads in full:

> This doctrine is antithetical to the very notion of quantification, the mainspring of modern logic. Quantification depends on there being values of variables, same or different absolutely; grant quantification and there remains no choice about identity, not for variables. For a language with quantification in it there is but one legitimate version of "$x = y$" (not counting equivalent versions). There is even a general criterion of whether a given open sentence in a given language provides the legitimate version of "$x = y$." [footnote: See my "Reply to Professor Marcus,"

Synthese 13 (1961), 325 ff.]

Simplifying somewhat, what Quine is saying here is that, if one augments any first-order theory with two binary relation symbols, say = and =′, together with the usual equality axioms formulated in terms of both = and =′, the extended theory proves $\forall x \forall y\,(x = y \leftrightarrow x =' y)$, so that there is no room for Geach's contention that objects may be identical in one sense but distinct in another.

That this is the right reading of the passage becomes quite clear if we consult the reference provided by Quine himself, which is his famous "Reply to Professor Marcus" from the Boston Colloquium. On pages 325–326, Quine argues that any two identity predicates are provably coextensive in any theory formulated in standard first-order logic, where by an identity predicate he understands any formula $\phi(x, y)$ in two free variables for which the theory proves the sentence $\forall x \phi(x, x)$ ("strong reflexivity") and all instances of the schema $\forall x \forall y(\phi(x, y) \wedge \theta(x) \rightarrow \theta(y))$ ("substitutivity").

This result establishes *uniqueness* ("no more than one"), not *existence* of an identity predicate, and we see that what Quine means in the 1964 piece when he speaks of quantification depending on the values of variables being "same or different absolutely" is just that it is impossible to combine standard quantification with two extensionally diverging identity predicates—identity *predicates*, mind you, which are object-linguistic expressions with certain provable properties relative to a formal theory. The *relation* of identity is not even the topic of Quine's discussion. There is thus no argument here whatsoever that quantification would be incomprehensible, or incoherent, in the absence of such a binary *relation* of numerical identity. If anything, quite to the contrary. In fact Quine notes (1961: 326)

> that there is no assurance, given a theory with recognized notations for quantification and the truth functions, that there is an identity predicate in it. It can happen that no open sentence in '*x*' and '*y*' is strongly reflexive and substitutive,

and further, on the same page,

> that if an open sentence in '*x*' and '*y*' does meet these two requirements, we may still find it to be broader than true identity when we interpret it in the light of prior interpretations of the primitive predicates of the theory.

Thus quantification, according to Quine, is perfectly possible in the absence of an identity predicate, and even if there is an identity predicate in the language at hand, its semantic value need not be the relation of numerical identity. Since Quine has, at best, established a relationship between quantification and identity *predicates*, nothing follows from his reasoning concerning the relationship between quantification and the identity *relation*. There is, therefore, no support of any kind in Quine's argument for the claim Hawthorne tries to bolster with its help.

Appendix A

Here we establish the result that first-order logic with identity but without variable recurrence within the scope of a quantifier is strictly less expressive than full first-order logic with identity.

To see this, consider a first-order language $\mathcal{L}$ with identity whose signature is given by two unary predicate symbols F and G. The recurrence-free $\mathcal{L}$-formulas can be defined as follows: Whenever x and y are distinct variables, Fx, Gx, and $x = y$ are recurrence-free $\mathcal{L}$-formulas; whenever ϕ is a recurrence-free $\mathcal{L}$-formula, so is $\neg\phi$; whenever ϕ and ψ are recurrence-free $\mathcal{L}$-formulas such that no variable occurs free in both ϕ and ψ, $(\phi \wedge \psi)$ is a recurrence-free $\mathcal{L}$-formula; and whenever x is any variable and ϕ a recurrence-free $\mathcal{L}$-formula, $\exists x\phi$ is a recurrence-free $\mathcal{L}$-formula.

Now let $\mathfrak{M}_0$ be the model for $\mathcal{L}$ whose domain is given by the set $M = \{0, 1\}$ and that interprets both F and G as the singleton set $\{0\}$. Let $\mathfrak{M}_1$ be the model for $\mathcal{L}$ whose domain is also the set M and that interprets F as $\{0\}$ (just like $\mathfrak{M}_0$), but interprets G as $\{1\}$. Clearly $\mathfrak{M}_0$ and $\mathfrak{M}_1$ can be distinguished by means of the non-recurrence-free $\mathcal{L}$-sentence $\exists x\,(Fx \wedge Gx)$, i.e. (2) in the main text, which is true in $\mathfrak{M}_0$ but false in $\mathfrak{M}_1$. However, as we will now prove, $\mathfrak{M}_0$ and $\mathfrak{M}_1$ make exactly the same recurrence-free $\mathcal{L}$-sentences true. It follows that the expressive power of first-order logic with identity is indeed diminished by disallowing variable recurrence. Hence the identity relation cannot subsume the semantic function of variable recurrence.

To obtain the desired result, we must prove a slightly more general fact:

Lemma *For every recurrence-free $\mathcal{L}$-formula ϕ there is a set A_ϕ included in the set $\mathsf{FV}(\phi)$ of variables occurring free in ϕ such that, for all variable assignments σ in the set M, σ satisfies ϕ in $\mathfrak{M}_0$ if and only if δ^ϕ_σ satisfies ϕ in $\mathfrak{M}_1$, where δ^ϕ_σ is the variable assignment that maps a variable x to $\sigma(x)$ if $x \notin A_\phi$, but to $1 - \sigma(x)$ if $x \in A_\phi$.*

The lemma can be proved by induction on the recurrence-free formula ϕ. If ϕ is atomic, it is of one of the forms Fx, Gx, and $x = y$. We let A_{Fx} and $A_{x=y}$ be the empty set, so that $\delta_\sigma^{Fx} = \delta_\sigma^{x=y} = \sigma$ for every assignment σ. This has the desired result, since the interpretation of F is the same in $\mathfrak{M}_0$ and $\mathfrak{M}_1$, and likewise for $=$. For the remaining atomic case, we let A_{Gx} be $\{x\}$, so that $\delta_\sigma^{Gx}(x) = 1 - \sigma(x)$ for every assignment σ. Since i is in the interpretation of G in $\mathfrak{M}_0$ if and only if $1 - i$ is in the interpretation of G in $\mathfrak{M}_1$, this too has the desired result.

If ϕ is compound, it is of one of the forms $\neg\psi$, $\exists x\psi$, and $(\psi \wedge \theta)$. We let $A_{\neg\psi}$ be A_ψ, so that $\delta_\sigma^{\neg\psi}$ is δ_σ^{ψ}. The induction hypothesis then immediately gives the desired result.

Let $A_{\exists x\psi}$ be $A_\psi \setminus \{x\}$, so that, for $y \in \mathsf{FV}(\exists x\psi)$, $\delta_\sigma^{\exists x\psi}(y)$ is $\delta_\sigma^{\psi}(y)$. For assignments τ and elements $i \in M$, let us write $\tau\{x := i\}$ for the x-variant of τ that maps x to i. We note that the existence of an $i \in M$ such that $\delta^{\psi}_{\sigma\{x:=i\}}$ satisfies ψ in $\mathfrak{M}_1$ is equivalent to the existence of a $j \in M$ such that $\delta_\sigma^{\psi}\{x := j\}$ satisfies ψ in $\mathfrak{M}_1$. This is because, if $x \notin A_\psi$, $\delta^{\psi}_{\sigma\{x:=i\}}$ is $\delta_\sigma^{\psi}\{x := i\}$ for each $i \in M$, while if $x \in A_\psi$, $\delta^{\psi}_{\sigma\{x:=i\}}$ is $\delta_\sigma^{\psi}\{x := 1 - i\}$. Further, for any $j \in M$, $\delta_\sigma^{\psi}\{x := j\}$ is the same function as $\delta_\sigma^{\exists x\psi}\{x := j\}$, since δ_σ^{ψ} and $\delta_\sigma^{\exists x\psi}$ differ at most in what they assign to x. With these observations in place, we see that σ satisfies $\exists x\,\psi$ in $\mathfrak{M}_0$ if and only if for some $i \in M$, $\sigma\{x := i\}$ satisfies ψ in $\mathfrak{M}_0$, if and only if (by induction hypothesis) for some $i \in M$, $\delta^{\psi}_{\sigma\{x:=i\}}$ satisfies ψ in $\mathfrak{M}_1$, if and only if for some $j \in M$, $\delta_\sigma^{\psi}\{x := j\}$ satisfies ψ in $\mathfrak{M}_1$, if and only if for some $j \in M$, $\delta_\sigma^{\exists x\psi}\{x := j\}$ satisfies ψ in $\mathfrak{M}_1$, if and only if $\delta_\sigma^{\exists x\psi}$ satisfies $\exists x\,\psi$ in $\mathfrak{M}_1$, as desired.

Finally, let $A_{(\psi\wedge\theta)}$ be $A_\psi \cup A_\theta$. We note that δ_σ^{ψ} and $\delta_\sigma^{(\psi\wedge\theta)}$ agree on

$\mathsf{FV}(\psi)$, and δ_σ^θ and $\delta_\sigma^{(\psi\wedge\theta)}$ agree on $\mathsf{FV}(\theta)$. To see the former, observe that for $x \in \mathsf{FV}(\psi)$, $x \in A_\psi \cup A_\theta$ if and only if $x \in A_\psi$, since $(\psi \wedge \theta)$ is recurrence-free, so that ψ and θ do not share any free variables, and x cannot be in $A_\theta \subseteq \mathsf{FV}(\theta)$. The corresponding claim for θ follows analogously. We then have that σ satisfies $(\psi \wedge \theta)$ in $\mathfrak{M}_0$ if and only if σ satisfies both ψ and θ in $\mathfrak{M}_0$, if and only if (by induction hypothesis) δ_σ^ψ satisfies ψ in $\mathfrak{M}_1$ and δ_σ^θ satisfies θ in $\mathfrak{M}_1$. By the fact just observed, and since satisfaction depends only upon what is assigned to the variables actually occurring free in a formula, this is the case if and only if $\delta_\sigma^{(\psi\wedge\theta)}$ satisfies ψ in $\mathfrak{M}_1$ and $\delta_\sigma^{(\psi\wedge\theta)}$ satisfies θ in $\mathfrak{M}_1$, i.e. if and only if $\delta_\sigma^{(\psi\wedge\theta)}$ satisfies $(\psi \wedge \theta)$ in $\mathfrak{M}_1$, as desired. This completes the proof of our lemma.

The result concerning recurrence-free *sentences* now follows from the lemma by observing that a sentence (closed formula) is true in a model if and only if it is satisfied by at least one assignment. Thus a recurrence-free sentence ϕ is true in $\mathfrak{M}_0$ if and only if there is an assignment σ that satisfies ϕ in $\mathfrak{M}_0$; by the lemma, this is the case if and only if there is an assignment (to wit, δ_σ^ϕ, which in the case of sentences just is σ) that satisfies ϕ in $\mathfrak{M}_1$, if and only if ϕ is true in $\mathfrak{M}_1$.[8]

In an analogous fashion, one can prove the corresponding result about

[8] It follows from results by Monk (1965) that, in first-order languages with identity, we *can* dispense with variable recurrence within *atomic* formulas, that is, the fragment of such a language in which primitive predicates must be followed by strings of mutually distinct variables is equi-expressive with the full language. By way of example, consider the formula $\exists x\, Rxx$, which is logically equivalent to $\exists x\, \exists y(Rxy \wedge x = y)$. Of course this paraphrase depends heavily on variable recurrence *across* atomic formulas. Thanks to Lloyd Humberstone for bringing this result of Monk's to my attention.

first-order languages in predicate functor notation: PFL-languages with an identity predicate but without the functor Ref are, in general, strictly less expressive than the same languages with Ref.

Appendix B

Here we recall the syntax and semantics of predicate functor logic.

A language $\mathcal{L}$ of predicate functor logic with identity is characterized by its set of non-logical primitive predicates, each of which has an arity $n \geq 0$. The primitive symbols of such a language $\mathcal{L}$ are, in addition to its non-logical primitive predicates, the functors Der, Ref, Inv, inv, and Neg, all of arity 1, the functor $\times$ of arity 2, and the identity predicate I of arity 2 (as well as left and right parentheses for grouping).

The predicates of $\mathcal{L}$ and their arities are defined inductively as follows. Every n-ary non-logical primitive predicate P of $\mathcal{L}$ is an $\mathcal{L}$-predicate of arity n. The identity predicate I is an $\mathcal{L}$-predicate of arity 2. If ϕ is an n-ary $\mathcal{L}$-predicate, Derϕ is an $(n-1)$-ary $\mathcal{L}$-predicate (except when $n = 0$, in which case Derϕ is also 0-ary), Refϕ is an $(n-1)$-ary $\mathcal{L}$-predicate (except when $n = 0$, in which case Refϕ is also 0-ary), and Invϕ, invϕ, as well as Negϕ, are n-ary $\mathcal{L}$-predicates. If ϕ is an n-ary, and ψ an m-ary, $\mathcal{L}$-predicate, $(\phi \times \psi)$ is an $(n+m)$-ary $\mathcal{L}$-predicate. The 0-ary $\mathcal{L}$-predicates are also called *$\mathcal{L}$-sentences*.

A model $\mathfrak{M}$ for $\mathcal{L}$ is a pair $(M, \mathcal{I})$, where M, the *domain* of $\mathfrak{M}$, is a non-empty set and $\mathcal{I}$ is a function assigning to each n-ary primitive predicate P of $\mathcal{L}$ an n-ary function $P^{\mathfrak{M}}$ from M^n to the set $\{0, 1\}$ of truth values. M^0 is the singleton set containing the empty tuple $\langle\rangle$ as a member, so a function f

from M^0 to $\{0,1\}$ maps $\langle\rangle$ to $f(\langle\rangle) \in \{0,1\}$, and we will simply identify such f with $f(\langle\rangle)$.

For each n-ary $\mathcal{L}$-predicate ϕ, we recursively define the interpretation $\phi^{\mathfrak{M}} : M^n \to \{0,1\}$ of ϕ in $\mathfrak{M}$ as follows: Where P is an n-ary primitive non-logical predicate of $\mathcal{L}$, $P^{\mathfrak{M}}$ is already given by $\mathfrak{M}$. The interpretation $I^{\mathfrak{M}}$ of the identity predicate I in $\mathfrak{M}$ is true identity, i.e. the binary function on M that maps each pair (a,a) to 1 and every other pair to 0. Where ϕ is an $(n+1)$-ary $\mathcal{L}$-predicate, $(\mathsf{Der}\phi)^{\mathfrak{M}}$ is the function that maps any n-tuple $(a_1,\ldots,a_n)$ to $\max\{\phi^{\mathfrak{M}}(a_1,\ldots,a_n,a) \,|\, a \in M\}$; if ϕ is 0-ary, $(\mathsf{Der}\phi)^{\mathfrak{M}}$ is just $\phi^{\mathfrak{M}}$. Where ϕ is an $(n+1)$-ary $\mathcal{L}$-predicate, $(\mathsf{Ref}\phi)^{\mathfrak{M}}$ is the function that maps any n-tuple $(a_1,\ldots,a_n)$ to $\phi^{\mathfrak{M}}(a_1,\ldots,a_n,a_n)$; if ϕ is 0-ary, $(\mathsf{Ref}\phi)^{\mathfrak{M}}$ is just $\phi^{\mathfrak{M}}$. If ϕ is an n-ary $\mathcal{L}$-predicate, $(\mathsf{Inv}\phi)^{\mathfrak{M}}$ is the function that maps any n-tuple $(a_1,\ldots,a_n)$ to $\phi^{\mathfrak{M}}(a_n,a_1,\ldots,a_{n-1})$, $(\mathsf{inv}\phi)^{\mathfrak{M}}$ maps any n-tuple $(a_1,\ldots,a_n)$ to $\phi^{\mathfrak{M}}(a_1,\ldots,a_{n-2},a_n,a_{n-1})$, and $(\mathsf{Neg}\phi)^{\mathfrak{M}}$ maps any n-tuple $(a_1,\ldots,a_n)$ to $1-\phi^{\mathfrak{M}}(a_1,\ldots,a_n)$. Finally, if ϕ is an n-ary and ψ an m-ary $\mathcal{L}$-predicate, $(\phi \times \psi)^{\mathfrak{M}}$ maps any $(n+m)$-tuple $(a_1,\ldots,a_n,b_1,\ldots,b_m)$ to $\min\{\phi^{\mathfrak{M}}(a_1,\ldots,a_n), \psi^{\mathfrak{M}}(b_1,\ldots,b_m)\}$.

If ϕ is a sentence of $\mathcal{L}$, we say that ϕ is *true in* $\mathfrak{M}$ just in case $\phi^{\mathfrak{M}} = 1$; otherwise the sentence ϕ is *false in* $\mathfrak{M}$.

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