

Inquisitive Semantics and Logic

MSc Thesis (*Afstudeerscriptie*)

written by

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Chapter 1

Inquisitive Semantics

This chapter introduces inquisitive semantics as a research program for the study of the semantics of natural language questions and disjunctions. I will begin by providing a brief outline of the necessary background on question semantics and erotetic logics, and I will then present inquisitive semantics, by first laying out its main tenets and then defining a propositional logical system, InqL , that instantiates the inquisitive program.

1.1 Preliminaries

Since Hamblin’s (1958) seminal paper on the semantics of questions, most semanticists and philosophers have come to agree that “knowing what counts as an answer is equivalent to knowing the question,” (Hamblin, 1958) that is, the semantic content of a question must give its answerhood conditions. One standard way to implement this idea is to identify the meaning of a question with a set of propositions, namely the set of all propositions that are possible answers to that question (the earliest example of such a semantics can be found in Hamblin, 1973). Probably the most successful implementation of this intuition is due to Groenendijk and Stokhof (1984), who take the *sense* of a matrix question to be the set of all its possible mutually exclusive answers, partitioning the logical space. The partition approach will serve as a point of departure in what follows.¹

1.1.1 A partition theory of questions

Within the partition framework, it is customary to define a query logic over a standard assertive language,² by means of a question-forming operator that applies only at the topmost level, never occurring in embedded subformulas. Thus:

¹The formulation of the partition theory that I will present is mostly based on Groenendijk (1999), or rather a straightforward adaptation of that system to a *propositional* query language.

²By abuse of terminology, I will for the most part of this thesis use the term ‘language’ to refer to a set of well-formed formulas. Whenever I use it to mean a set of symbols from which formulas are constructed I will explicitly say so.

Definition 1 (Classical propositional query language). Let L be a language of propositional logic. QL , a classical propositional query language, is the smallest set such that, for each $\varphi \in L$, $\varphi \in QL$ and $?\varphi \in QL$. \dashv

The language QL contains such formulas³ as p , $?p$, $?(p \wedge q)$, $?(p \rightarrow q \vee (r \wedge s))$, but not $p \wedge ?q$, $p \rightarrow ?q$, $\neg\neg\neg?p \rightarrow ??r$, for obvious reasons. The interrogative formulas $?\varphi$ that indeed are a part of this language are to be interpreted as polar questions that partition the logical space, in a manner I'll make explicit shortly, so as to contain only the two logical cells that correspond to the assertive formulas (answers) φ and $\neg\varphi$, or “yes” and “no.”

Models for a classical propositional query language are an extension of possible worlds models for declarative semantics. Specifically, we will deal with models that consist of a relation between possible worlds, intended to model a notion of *indifference*, following Hulstijn (1997). Intuitively, two worlds will be connected just in case the difference between those two worlds is not at issue. For example, suppose we want to consider a model for the question

(1) Is it raining?

In our terms, a model for (1) cannot have a connection between worlds w_1 and w_2 when they disagree as to whether (2), the assertive sentence underlying (1), is true or not.

(2) It is raining.

That is, a model for (1) tells us that we are interested in what distinguishes worlds where (2) is true from those where it is false. We are however indifferent to all other issues, so connections will be present between, say, two worlds that agree as to whether (2) but disagree with respect to (3), which is not at issue.

(3) France is a monarchy.

Now, if we take this relation of indifference to be an equivalence relation, it follows from the partition theorem that it uniquely induces a partitioning of the underlying set of possible worlds it is built upon. Thus, the minimal model for (1) is the relation of indifference on an underlying set of possible worlds that contains all pairs of worlds except those where the two worlds disagree as to the value of (2). Furthermore, it induces a unique partition of the underlying set of worlds, namely, that partition which has two cells, one occupied by all worlds where (2) is true, and the other made out of all the worlds where (2) is false.

Definition 2 (Classical query models). A model for the classical query language QL is a reflexive, symmetric and transitive relation $\sigma \subseteq W \times W$, where W is the set of all total valuations on the set P of propositional atoms of QL . \dashv

The models of Definition 2 can be represented pictorially as in Figure 1.1, a model for the query language with only two propositional atoms, say p and q . Each circle represents a world in the basic set of possible worlds W on which

³Throughout this text, I will assume standard notational conventions, viz. left associativity and the scale of association of logical connectives whereby ‘ \neg ’ associates to the smallest subformula to its right, and ‘ \wedge ’ and ‘ \vee ’ take precedence over \rightarrow . That is, $\neg\varphi \vee \psi$ is an abbreviation of $((\neg\varphi) \vee \psi)$, and $\varphi \wedge \psi \rightarrow \theta \vee \chi$ an abbreviation of $((\varphi \wedge \psi) \rightarrow (\theta \vee \chi))$. The question unary connective ‘?’ default-associates in the same manner as negation.

the model proper is a relation; the left hand digit gives the truth value of p at that world and the right hand one that of q . The relation is represented by arrows connecting worlds. Notice that the model is, strictly speaking, only the relation, the arrows in this representation, that is, our model is the reflexive, symmetric and transitive closure of

$$\{\langle w_{11}, w_{10} \rangle, \langle w_{11}, w_{01} \rangle, \langle w_{11}, w_{00} \rangle, \langle w_{10}, w_{01} \rangle, \langle w_{10}, w_{00} \rangle, \langle w_{01}, w_{00} \rangle\} .$$

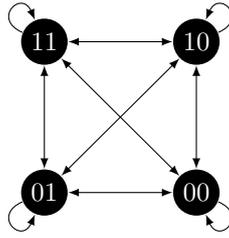


Figure 1.1: The indifferent, ignorant model for $P = \{p, q\}$

In this example, all reflexive pairs in $W \times W$ are present, indicating that all possibilities are open. I will call such a model an *ignorant* model. Moreover, every two distinct worlds are connected to one another, which I interpret to mean, as sketched above, that nothing is at issue in this model, that is, that the model is *indifferent*. Whenever a model is a total relation over the underlying set of worlds W , as in Figure 1.1, it is both ignorant and indifferent. Formally:

Definition 3 (Ignorance and indifference). For $\sigma \subseteq W \times W$ a model of a query language according to Definition 2, we say that σ is *ignorant* iff

$$(\forall w \in W) \langle w, w \rangle \in \sigma ,$$

and that σ is *indifferent* iff

$$(\forall \langle w, w \rangle, \langle w', w' \rangle \in \sigma) \langle w, w' \rangle \in \sigma . \quad \dashv$$

Remark 4. A model $\sigma \subseteq W \times W$ is ignorant and indifferent iff $\sigma = W \times W$. \dashv

We are now ready to give a semantics for the language QL . As the definition of an ignorant state above may have already hinted at, I find an update semantics (Veltman, 1990, 1996) that mirrors information growth by eliminating worlds from models to be especially perspicuous. Luckily, the basic intuitions that a Stalnakerian view of the common ground (or of an information state) gives us can be straightforwardly imported into the enriched model theory used here. Nothing in this thesis hinges on the choice of an update semantics formulation, in a sense to be made explicit in Chapter 2, although the remainder of the present chapter will exclusively refer to update semantics.

Definition 5 (Semantics for QL). For σ a classical query model as in Definition 2, the update of σ with a formula φ of QL , written $\sigma[\varphi]$, is inductively

defined as follows.

$$\begin{aligned}
\sigma[p] &= \{\langle i, j \rangle \in \sigma : i(p) = j(p) = 1\} \\
\sigma[\neg\varphi] &= \sigma - \sigma[\varphi] \\
\sigma[\varphi \wedge \psi] &= \sigma[\varphi] \cap \sigma[\psi] \\
\sigma[?\varphi] &= \{\langle i, j \rangle \in \sigma : \langle i, i \rangle \in \sigma[\varphi] \text{ iff } \langle j, j \rangle \in \sigma[\varphi]\} \quad \dashv
\end{aligned}$$

In words, the atomic clause above eliminates all pairs of worlds such that one or both worlds assign the value 0 (false) to p , and the question clause keeps only those pairs of worlds whose two elements⁴ are in sync with respect to passing or failing an update with φ . The conjunction and negation clauses are self explanatory, and disjunction and implication can be defined by standard abbreviations: $\varphi \vee \psi$ as $\neg(\neg\varphi \wedge \neg\psi)$ and $\varphi \rightarrow \psi$ as $\neg(\varphi \wedge \neg\psi)$.

This semantics can express the full gamut of classical, partition questions à la (propositional) Groenendijk and Stokhof (1984). To give just one example, Figure 1.2 represents the update of a simple σ as in Figure 1.1 with the formula $?p$. As mentioned earlier, because the indifference relation that our models represent is an equivalence relation, the partition theorem allows to go back and forth between the relation and the partition it induces. In Figure 1.2, I highlight the partition induced by the indifference relation by drawing a shape around each cell.

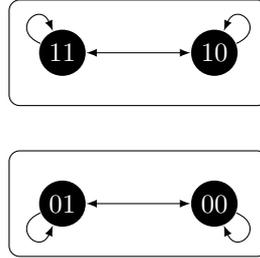


Figure 1.2: $\sigma[?p]$

1.1.2 Issues with the partition framework

The kind of partition theory instantiated above for the propositional case has been quite influential for the past twenty-five years;⁵ indeed, virtually all accounts of question semantics have assumed its basic tenets to be incontrovertible desiderata. In the paragraphs that follow, I will question the framework from three different fronts.

⁴Notice that, to see whether a world w makes a declarative sentence true or false we can simply look at whether the corresponding reflexive pair is in the update with that formula.

⁵But it is important to remark that the propositional case I am restricting this discussion to was never the focus of much attention due to its supposed triviality (Groenendijk and Stokhof, 1997, but see). This thesis will show however that a number of interesting questions arise even from considering only propositional logic.

Formal issues

All instantiations of the partition theory that I am familiar with, in fact most accounts of the logic of questions for that matter, entail a sharp syntactic and semantic distinction between declaratives and interrogatives. Consider for example the language QL of Definition 1. The definition of that language proceeded in two stages, first we defined (or rather assumed) a language of propositional logic, and then we added to that set the result of prefixing each of the standard propositional formulas with a ‘?’. It is therefore possible to distinguish between two subsets of QL , call them L and Q , respectively the declarative sentences of the language and the interrogative ones. This syntactic distinction is meaningful at the semantic level as well. The sentences in L have the potential to eliminate reflexive pairs of worlds in a model, while those of Q will eliminate non-reflexive pairs, i.e., connections between possible worlds. Crucially, no combination of these two kinds of semantic processes, providing information and raising issues, is possible with the language we are considering, as any given sentence of QL is either declarative or interrogative. This seems unwarranted:

- (4) Jane is a genius, but does she know it?
 $p \wedge ?q$

In (4), we have a natural language sentence that conjoins a declarative sentence and an interrogative one, to form a complex sentence that would most intuitively be formalized by a formula of the shape $p \wedge ?q$. Formulas such as this are entirely absent from QL .

Moreover, QL does not even allow, say, conjunction of interrogative sentences as in (5).

- (5) Is John coming to the party, and is Mary coming as well?
 $?p \wedge ?q$

By stipulating that the question operator can only apply at the topmost level in a formula, the language QL does not recognize such formulas as in (4) and (5) as well-formed. Now, one might argue that the kind of conjunction we see in (4) and (5) is of an importantly different kind than that of say (6), where two declaratives are conjoined.

- (6) John fell and Max pushed him.

The fact that there is no real need for a pause between conjuncts in (6), as opposed to (4) or (5), might be taken to be an indication that ‘and’ in (4) and (5) operates at a higher level than ‘and’ in (6), perhaps the discourse level. Be that as it may, a query language in the most abstract sense ought to be able to represent, respectively express, *hybrid* formulas, respectively meanings, that both provide information and raise issues. The question of how natural language expresses such meanings is a separate one, and it should inform us about how the query language *applies* to the study of natural language, what restrictions are in order, what level of meaning (term-level, sentence-level, discourse-level) is involved in the expression of individual meanings. It is therefore my position that the logical building blocks of a semantics of questions should allow the well-formedness of sentences whose meanings are interpretable, even when it may seem that such sentences do not have a direct correlate in a sentence of a

natural language.

Now, we could modify the syntax and semantics of QL in order to include sentences of the same form as (4) or (5) as well-formed and interpretable, in fact this is rather straightforward if we redefine conjunction as update sequencing or add a sequencing clause to the update definitions. The resulting language is certainly an improvement for the reasons I present above, but it is not enough. Consider the following example of a possible question in an (extremely) introductory set-theory exam.

- (7) Answer one of the following two questions.
- a. Is the collection of all infinite sets a set?
 - b. Is the Axiom of Choice equivalent to the Well-Ordering Theorem in ZF?

Intuitively, (7) offers two questions to the examinee, namely (7-a) and (7-b), and asks of the examinee that she choose one of these two questions and answer it. Crucially, the examiner will be satisfied with an answer to either question.

One intuitively compelling (propositional logic) formalization of the inquisitive discourse in (7) is the formula $?p \vee ?q$, which in fact has been used at least since Groenendijk and Stokhof (1984). Even if we change the syntax of QL so as to let this formula be well-formed, making it interpretable within the partition theory involves a sophisticated mechanism of lifting $?p$ and $?q$ to generalized quantifiers over questions and connecting the resulting generalized quantifiers via disjunction (see Groenendijk and Stokhof, 1984, 1989). While Groenendijk and Stokhof argue that these lifted meanings account for both matrix and complement disjunctions of questions, Szabolcsi (1997) makes a strong empirical and conceptual case against using the lifted objects in matrix contexts. I will return to this issue closer to the end of this chapter; for now, it suffices to note that QL 's heavily constrained syntax and semantics make it either impossible or non-trivially cumbersome to express the category of formulas that may be intuitive representations of discourses like the one in (7).

Another interesting consequence of this sharp formal distinction between declarative and interrogative sentences is that such query languages have a less than obvious proof-theory. To give just one example, if we take a standard update semantics definition of semantic entailment and assume the existence of a sound and complete proof system for it, it evidently lacks a deduction theorem.

Definition 6 (Support). A model σ *supports* a formula φ of QL , notated $\sigma \models \varphi$, iff $\sigma[\varphi] = \sigma$. ↯

Definition 7 (Entailment). For two formulas φ and ψ of QL , we say that φ entails ψ , in symbols $\varphi \models \psi$ iff for all models σ , $\sigma[\varphi] \models \psi$. ↯

Under the definitions above, and assuming the existence of a sound and complete turnstile relation, it is easy to see that $p \vdash ?p$, or more generally, a question is entailed by any of its answers. However, we do not have that $\vdash p \rightarrow ?p$, as one would expect from a standard logic with a deduction theorem, for the very simple reason that $p \rightarrow ?p$ is *not* a formula of QL . Lack of a deduction theorem is what made the axiomatization of Groenendijk's (1999) logic by ten Cate and Shan (2007) a more sophisticated endeavor than the typical axiomatization of tautologies: ten Cate and Shan (2007) give a sound

and complete axiomatization of the entailment relation of Groenendijk’s Logic of Interrogation (1999).

Notice that the proof-theory of erotetic logics is not a matter only of interest to the pure logician. Since at least Belnap and Steel (1976), having a formal theory of the relation between questions and answers and between questions and their subquestions has been an important desideratum of the erotetic logic enterprise. One compelling way to achieve that goal is to make sure entailment (i.e., \models) encodes the relations of answerhood and subquestionhood, besides (assertive) consequence, as done for example by Groenendijk (1999).

Ineffable meanings

Moving on to the adequacy of the partition theory to describe interrogative meanings, it is important to note that Groenendijk and Stokhof’s original work had already encountered meanings that seemed to go beyond partitions. From the realm of constituent questions, the case of mention-some questions is an especially well-known one. Questions such as (8) are most typically taken not to require a complete answer (in Groenendijk and Stokhof’s sense of ‘complete’), and answers to it are also typically interpreted in a non-exhaustive way, suggesting that the meaning of (8) is a set of non mutually exclusive answers.

- (8) Where can I buy Austrian newspapers?
At the Neue Galerie in the Upper East Side.

At the level of a propositional query language, an even more telling example can be found in the elusive case of conditional questions, as in (9).

- (9) If John comes to the party, will Mary come as well?

Under at least one reading (indeed, I would submit, the most salient reading)⁶ of (9), it is a polar question where the ‘yes’ answer states (10-a) and the ‘no’ answer (10-b).

- (10) a. If John comes to the party Mary will also come.
 $p \rightarrow q$
b. If John comes to the party Mary won’t come.
 $p \rightarrow \neg q$

There are a number of competing analyses of (9) on the market, the one I will adopt here was proposed by Velissaratou (2000) and used by Groenendijk (2007, 2008a). In a nutshell, Groenendijk argues that a conditional question such as (9) is most naturally translated into a formula like $p \rightarrow ?q$, and it corresponds to a meaning that distinguishes two possible answers, namely $p \rightarrow q$ and $p \rightarrow \neg q$, that are *not* mutually exclusive. Indeed, the two propositions these formulas correspond to *overlap*, in that they both contain the $\neg p$ worlds. Figure 1.3

⁶The other possible reading inquires about the existence of a certain connection, perhaps most naturally a causal one, between the antecedent and the consequent. In that reading, the ‘yes’ answer means the same as in the reading we are interested in, namely that $p \rightarrow q$, but the ‘no’ answer does not commit the responder to $p \rightarrow \neg q$, it is rather just the statement that there is no necessary, causal connection between p and q . A natural paraphrase of this ‘no’ answer is something like “No, it may well be that John comes to the party but Mary doesn’t, the two situations are just completely independent of each other.” See also footnote 19.

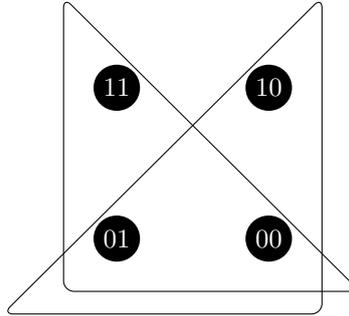


Figure 1.3: If p, q ?

highlights the two possible answers to the question “if p, q ?” in this approach.⁷

This proposal has the advantage of assigning the expected answerhood conditions to conditional questions without assuming more sophisticated mechanisms than those available in a standard propositional logic, but it implies a view of question meanings that accepts non-partitioning questions, therefore going beyond the expressive power of QL .

There are two main alternative proposals that try to maintain the partition picture. Hulstijn (1997) argues that the conditional question $p \rightarrow ?q$ has three possible answers, namely $p \wedge q$, $p \wedge \neg q$ and $\neg p$, an answer-set which partitions the logical space. He further analyzes answers of the sort $p \rightarrow q$ as partial answers to the conditional question. While his proposal succeeds in preserving the partition picture, it does so at a cost. Firstly, in keeping with the partition theory’s notions of answerhood, Hulstijn must say that the answers in (10) are only partial answers, i.e., that they somehow only partially resolve the issue raised by (9), which seems counterintuitive: after an utterance of (9), any of the answers in (10) seem to address the issue raised fully, not just partially. Secondly, it is a consequence of this proposal that the answers $p \wedge q$ and $p \wedge \neg q$ are complete answers to the question posed, which is not the case: a sentence like “John and Mary are coming to the party” certainly addresses the issue raised by (9), but it provides extra information, namely that John is coming to the party, which was not the subject of inquiry. This again goes against the notion of a complete answer in the partition theory, as the answer set is supposed to contain those answers that fully resolve the issue raised *and only* the issue raised. Thirdly, the answer $\neg p$ is predicted to be appropriate, also an undesirable result on empirical grounds. If someone were to respond to (9) with “John isn’t coming to the party,” he might be doing a number of things, such as signaling that the question is irrelevant or protesting against a possible presupposition of the question, but he would not be answering the question in any intuitive sense. Clearly, it is important to distinguish the status of $\neg p$ from that of answers to the question, complete or partial, and Hulstijn’s proposal can

⁷The reader will have no doubt noticed that I’m using \rightarrow to represent natural language *if...then*. While we know that this is an oversimplification, I will ask for some suspension of disbelief on the grounds that 1. material implication can capture *some* of the properties of English *if...then* and, more importantly, 2. part of the objective of this thesis is to explore *propositional* erotetic systems; as such, the tools of modal logic or of truly dynamic update semantics, necessary to give a more adequate semantics of natural language conditionals, are not at my disposal.

only do so via some superimposed pragmatic mechanism that excludes $\neg p$ as an answer to (9).

Isaacs and Rawlins (2008) explore a different line, proposing that conditional questions be analyzed in a properly dynamic, stack-based update system. Informally, they define an update with *if* φ , ψ that does the following. First, a copy of the state being updated is created and put on top of it, forming (or adding to) a stack. Then, that state is updated with the antecedent, and finally the resulting state is updated with the consequent, in the case of conditional questions, a question. This proposal is still within the partition framework: the update with $?q$ is performed on the topmost state in the stack which has already been updated with the antecedent p , so the inconvenient $\neg p$ -worlds are not in that state, which is therefore partitioned by $?q$. An affirmative answer to the question percolates down to the original state, discharging the temporary p -state, and resulting in an update of the original state with $p \rightarrow q$, as intended. Similarly for the negative answer.

This proposal is superior to Hulstijn’s in that it delivers the desired answerhood conditions, but it achieves that goal by resorting to the expressive power of a full dynamic system. Dynamic systems are of course nothing to be afraid of *per se*, but they add significant complexity to an analysis, a move that must be adequately motivated. Groenendijk’s and my contention is that, in this particular case, a dynamic system is not required, given that the desired answerhood conditions become expressible in a perfectly static system if we just drop the partition requirement, one I argue should be dropped for independent reasons as well, as the present thesis hopes to show.

As demonstrated by Figure 1.3, conditional questions can be given a very simple, static semantics if we are allowed to express meanings that *do not* correspond to partitions. Such meanings are inexpressible in the query system QL or any of its partition theory cousins, pace Isaacs and Rawlings’s (2008) dynamic system.⁸

In later sections of this thesis I will discuss other classes of meanings that cannot be expressed by partitions, namely alternative questions and a certain use of (declarative) natural language ‘or’.

⁸As discussed in footnote 7, the simple propositional logics I am considering lack the tools to express strict implications and other more sophisticated notions. It is however interesting to note that perhaps a theory of conditionals such as the one introduced by Stalnaker (1968) can also preserve the partition framework while yielding the right answerhood conditions, as Daniel Rothschild pointed out to me. Speaking informally, $\llbracket \textit{if } p, q \rrbracket$ can be seen as an update function that keeps in a state all those worlds w such that the closest p -world to w is a q -world, assuming some appropriate extension of the update partition system QL that can encode an accessibility relation, and under a presupposed similarity ordering. Similarly for $\llbracket \textit{if } p, \neg q \rrbracket$. Now, these two sets correspond to the Stalnaker conditionals of the answers in (10) but, contrary to what happens if we consider material implication in a static system, the two sets of worlds are disjoint. Suppose there is some w that is in both, then w must be a world such that its closest p -world is both a q -world and a $\neg q$ -world, which is a contradiction. Therefore, the Stalnaker conditionals that correspond to the answers in (10) are disjoint and form a partition of logical space.

Let me reiterate my position on this matter. While granting that many if not most uses of natural language conditional constructions express meanings that material implication fails to capture and that might necessitate a modal analysis, I am for the purposes of this thesis interested in exploring the simplest propositional logic and discovering just how far we can take it, trying to express question meanings. I therefore leave both a careful development of a modal analysis of conditional questions and a discussion of its merits and shortcomings to future work.

Why partitions?

The strongest empirical argument for the partition theory comes from the realm of constituent *wh*-complements. Groenendijk and Stokhof (1982, 1984) argue that, in light of the inconsistency of statements like (11), an embedded question must denote its true and exhaustive answer, which has as a consequence that matrix questions correspond to partitions.

(11) #John knows who came to the party, but he's not sure if Jane did.

Even if we grant that this is in fact always the case with embedded constituent questions, it is less clearly so with matrix questions. Surely, (13-a), with a focus intonation, must be interpreted as an exhaustive answer to (12), but I find it at least arguable whether (13-b), with no special intonation, or the non-fragment answers in (13-c) and (13-d), are necessarily, or even most naturally, interpreted exhaustively.

(12) Who came to the party?

- (13) a. [Mary]_F.
b. Mary.
c. Mary came to the party.
d. Mary did.

The partition theorist's only possible reply to this observation is that the answers in (13-b)–(13-d) are partial answers, the same applying to the non-exhaustive question-answer pair in (8). Now, if the notion of a partial answer is taken to be purely technical and defined simply as a non-exhaustive answer, then the partition theory's contention becomes irrefutable: exhaustive answers will be the ones our semantics produces, and partial answers will be acceptable in some cases. If however we believe that partial-answerhood should capture an *intuitive* notion, then the partition theorist's reaction to the cases above becomes much less convincing, for the simple reason that, both in (8) and (13), the answers are perfectly felicitous, addressing the question asked, and, at least in (8), resolve it completely. Calling these “partial answers” seems like a dubious move.

If these observations are correct, they suggest that, while a partition theory may be the right way to analyze constituent complement questions, exhaustivity seems too strong a requirement for matrix questions.

Moreover, recall that the partition theory required us (in fact Groenendijk, 1999) to stipulate that the indifference relation captured by the query models be an equivalence relation. This meant stipulating that indifference is reflexive, symmetric and transitive. Now, from a purely conceptual point of view, it is easy to see why a relation of indifference ought to be reflexive and symmetric. We cannot possibly be interested in the difference between a world and itself — there is none — and, if we are not interested in the difference between *w* and *v*, then we cannot be interested in the difference between *v* and *w* either. The requirement of transitivity, however, is much less intuitive.

Indeed, it seems necessary, at least conceptually, to grant that it is possible for us to be indifferent with respect to how *w* differs from *v* and how *v* differs from *u*, but *not* to be indifferent regarding how *w* differs from *u*. The difference between *w* and *u* may well be big enough for us to be interested in it.

Interestingly, the partition theory of questions more or less tacitly underlies Lewis’s work on *relatedness* and *subject matter*. In particular, Lewis (1988) asks us to think of subject matter (e.g., the subject matter of a conversation, or even just of an assertion in the shape of single sentence), *equivalently* as one of the following:

- (14) a. A part of the world in intension,
- b. an equivalence relation between possible worlds,
- c. a partition, and
- d. a question.

Lewis is the first to admit that (14-a) is an elusive notion, so let us disregard it without much compunction. Although Lewis does not give an intuitive gloss of the equivalence relation (14-b), a natural candidate seems to be Hulstijn’s indifference relation. The same argument I offered above against transitivity can therefore be made in this context. Clearly, there is good reason to assume that (14-b) and (14-c) are stipulative requirements that lack conceptual motivation; in addition, I have at the very least cast doubt on whether (14-d), natural language questions, provide a good motivation for partitions. It seem therefore that a revision of Lewis’s definition of relatedness might be in order, and it would be interesting to see what consequences that shift might have.

In summation, transitivity of the indifference relation our query models try to capture is unwarranted from a conceptual perspective, and perhaps should be dropped. If we do that, however, we cease to have an equivalence relation, and partitions are no longer guaranteed to exist.

1.2 Inquisitive Semantics

Inquisitive semantics is a reaction to the issues raised above. First, it makes no syntactic distinction between declaratives and interrogatives, defining a powerful language that can express a full gamut of hybrid sentences that both provide information and raise issues; this makes the logic that corresponds to it a simple system, with most expected logical properties. Second, inquisitive semantics is a strictly more expressive system than a partition-based one, providing the straightforward account of conditional questions sketched above, as well as a whole new class of meanings that, I will argue, are needed to model certain natural language meanings. Third, it takes Hulstijn’s (1997) compelling idea of interpreting query languages in structured models that represent a relation of *indifference* and makes it conceptually sounder, by dropping the requirement of transitivity and thereby that of partitions.

1.2.1 An inquisitive program — questions meet disjunctions

Hamblin’s (1958) intuition of what a question should mean can be paraphrased as follows.

- (15) A question introduces a number of *alternatives* (its possible answers) and requires that one of them be chosen.

This idea is strikingly similar to the way Grice (1989) addresses natural language (or at least English) ‘or’:

A standard (if not *the* [PG’s emphasis] standard) employment of ‘or’ is in the specification of possibilities (one of which is supposed by the speaker to be realized, although he does not know which one).

That is, both questions and disjunctions raise possibilities, or alternatives, and convey ignorance of the speaker as to which one is the case. Indeed, it seems that ‘or’ might even share with questions the requirement that the issue of which of the alternatives is the case be addressed. For example, the dialog below is perfectly natural:

- (16) A: John or Mary will come to the party tonight.
B: Well, John is sick, so I guess it’s Mary that’s coming.

In (16), B is clearly not going off on a tangent by addressing the issue of which one of John and Mary will come to the party, in fact B is perceived as being rather cooperative: B understood A’s statement as expressing (among other things) ignorance and interest, and addressed it as though it implied the question “Who will come to the party, John or Mary?”

I will call the sense in which questions and disjunctions are similar their *issue-raising potential*, and will assume it to be located in the semantics instead of the pragmatics. This is by no means a trivial move, but it is justified by the fact that the alternatives put forth by both questions and disjunctions have semantic import, combined with the following postulate of inquisitive semantics:

- (17) Semantic alternatives are a result of the linguistic mechanism of raising issues.

That alternatives play a role in the *semantics* of questions was part of Hamblin’s (1958) postulates, and it is by now part of the consensus among semanticists. As for disjunctions, recent literature on alternative semantics that deal with, among other linguistic phenomena, free choice and counterfactual sentences with disjunctive antecedents have argued very convincingly for the need for semantic alternatives generated by natural language disjunctions.⁹

The postulate in (17) proposes that we relate the alternative-generating power of questions and disjunctions to their issue-raising potential, illustrated in the example above for disjunction, incontrovertibly present in questions.¹⁰

If these observations are on the right track, then the meanings of questions and disjunctions ought to be to a visible extent similar.

The main postulate of inquisitive semantics says that we should take this similarity to its direst consequences, by assuming that questions are, at a fundamental level of semantic analysis, disjunctions. Specifically, in the inquisitive propositional language to be defined in a few paragraphs, I will use the abbre-

⁹But see Chemla (2009) for arguments against a semantic treatment of such phenomena.

¹⁰There are at least two other alternative-generating classes of elements in natural language, namely indefinites (Kratzer and Shimoyama, 2002) and focused constituents (Rooth, 1985). I will briefly mention indefinites later in the very last section of this chapter, arguing that they can be seamlessly integrated into the program of inquisitive semantics. As for focus, it has been discussed in connection with inquisitive semantics by Balogh (2009).

viation

$$?\varphi := \varphi \vee \neg\varphi .$$

Although this simple move,¹¹ as the sections to come will show, allows us to address all the issues raised above against the partition theory, it deserves some immediate clarification.

The central claim is that, at the deepest level of semantic meaning, questions and disjunctions make use of the same semantic mechanism, that of introducing alternatives. This mechanism, I further argue, originates from natural language ‘or’ and is used by natural language interrogatives, whence my defining a question operator in terms of basic disjunction, and not the other way around.

One way to materialize this insight in our semantics would be to keep it in the meta-level: I could give definitions of disjunction and interrogation that involve the same mechanism at the meta-level, but that instantiate that mechanism differently. The inquisitive semantics and logic I define in this thesis however goes beyond that.

More than just claiming that the alternative-generating mechanism behind questions and disjunctions is the same, inquisitive semantics proposes that this insight be made explicit at the object level. That is, I propose that we define questions *in terms of* disjunctions, as per the abbreviation above. This means that I am taking the similarity between questions and disjunctions to mean quite literally that one class of meanings is derived from the other.

This may seem quite radical. Surely, natural language interrogative sentences are very different from declarative disjunctions, their syntax is different, their intonation patterns are different, and their uses, if not also their meaning, are not identical. The inquisitive semantics and logic instantiated in this thesis, to the extent that it overlooks those differences, is most likely an overly radical idealization, but it is one that, I will show, addresses all the issues raised above while incorporating an insight into the common properties of questions and disjunctions in its most literal interpretation. The differences that exist between natural language interrogative sentences and natural language disjunctive sentences do not invalidate this inquisitive program, they just suggest that further refinements will almost certainly be in order, surely at the syntactic and pragmatic levels, possibly also at a semantic level. It may well turn out that the significance of the similarities pointed out above must be obscured at the object-language level, because of the possible need to differentiate the semantics of interrogation and disjunction to a point where, in the object level definitions, the relation between the two is no longer visible. My stance on that matter is simple: I will begin by investigating the idealized system where the similarity between questions and disjunctions is taken to be identity, or rather inter-definability, and I will leave inquiry on why that is an overly radical hypothesis and how it should be weakened to future research. At the very least, the inquisitive semantics and logic studied here will give insights on the relation

¹¹Probably because it is so simple, it is not exactly unprecedented. Harrah (1961) argues that “a logic of questions, sufficient for the question-and-answer process, already exists within the logic of statements.” He proposes the abbreviation $F?$ for $F \vee \neg F$, and defines a direct answer to a question $F?$ as any of the disjuncts of $F?$. Harrah’s idea was dismissed by Hamblin (1963) as falling in the general category of theories that try to reduce questions to assertions, an enterprise Hamblin found completely misguided. Although I agree with Hamblin regarding the need for a true erotetic semantics, it is not as obvious to me as it was to him that Harrah’s research program was incompatible with that basic desideratum.

between questions and disjunction that will inform more linguistically tenable analyses of natural languages.¹²

Moreover, even if it turns out that the level of analysis at which the semantics of natural language interrogatives and disjunctions is the same is so deep as to become almost invisible at the surface level, and only of potential explanatory interest from an historical, diachronic perspective on language, the gain in logical tractability and expressive power that we get from the inquisitive logic I define here makes it worth recognizing and taking seriously what questions and disjunctions share.

The central claim of inquisitive semantics has linguistic motivation independent of the semantic and pragmatic parallels I introduced above. It is well known that very many natural languages share the same morpheme (or close variations thereof) for ‘or’ and question particles. Malayalam *oo* (Jayaseelan, 2004, 2008) is a good example:

(18) John-oo Bill-oo wannu.
John-or Bill-or came
‘John or Bill came.’

(19) Mary wannu-oo?
Mary came-or
‘Did Mary come?’

This is a very robust linguistic fact, true of languages like Japanese (*ka*), Korean (*na*) and a number of Slavic languages (*li*). To some extent, such familiar languages as Dutch and English also instantiate this morphological generalization. In Dutch, the embedded question complementizer is identical to the word for ‘or’, as shown below, and clearly English ‘whether’ is ‘wh’ + ‘either’, a disjunction morpheme.

(20) Ik heb Anne of Marie gezien.
I have Anne or Marie seen
‘I saw Anne or Marie.’

(21) Ik weet niet of Anne komt.
I know not or Anne comes
‘I don’t know if Anne is coming.’

Inquisitive semantics has the potential to straightforwardly account for this observation, since it relates questions and disjunctions in its object language. In other words, taking these linguistic data at their face value most naturally induces, at least as an initial working hypothesis, a semantic move of the sort made in inquisitive semantics.¹³

In the next section, I will define a minimal instantiation of the inquisitive program, the system *lnqL*, and show how it addresses the issues raised in the previous sections.

¹²These paragraphs benefited greatly from a discussion of a closely related topic with Anna Szabolcsi, for which I’m very grateful.

¹³It is important to keep in mind though that the inquisitive logic defined here is rather unsuited for sophisticated semantic work, the main reason for that being that this thesis concentrates exclusively on *propositional* inquisitive logic. See however AnderBois (2009) for an analysis of Yukatek Mayan questions and disjunctions in terms of the logic defined in this thesis.

1.2.2 The inquisitive system InqL

I begin by defining the languages of InqL.

Definition 8 (Inquisitive syntax). A language L_P of InqL, indexed to a finite set of propositional atoms P , is the smallest set such that

$$\begin{aligned} p &\in L_P \text{ for all propositional atoms } p \in P \\ \perp &\in L_P \\ \text{if } \varphi, \psi &\in L_P, \text{ then } \varphi \wedge \psi \in L_P \\ &\varphi \vee \psi \in L_P \\ &\varphi \rightarrow \psi \in L_P \end{aligned}$$

Following standard practice, I will use the abbreviations $\neg\varphi := \varphi \rightarrow \perp$ and $\top := \neg\perp$. +

Notice that I am constraining the set of propositional atoms for any given inquisitive language to a finite set. This means that there are infinitely many inquisitive languages, one for each finite set of propositional atoms. I will postpone the discussion of why I make this stipulation to Section 2.1.2; for the time being, suffice it to say that it serves the purpose of keeping the semantics manageable. At any rate, the restriction is of no consequence to the erotetic expressive power of the language.

Also in connection with Definition 8, notice that I define a perfectly standard propositional language, with negation in terms of implication and *falsum* as is common practice. There is no primitive question operator, rather, it will be defined as an abbreviation, as I suggested above.

InqL will be interpreted in models much like those defined for the partition query language QL above, except for the fact that InqL models need not be transitive relations. Thus:

Definition 9 (Inquisitive models). Models for InqL are relations $\sigma \subseteq W \times W$, where W is the set of total valuations (worlds) on the set of propositional atoms P of an inquisitive language L_P . σ s are in addition required to be reflexive and symmetric (NB, not necessarily transitive). We will call the set of all such states Σ_P , for each language L_P . +

Each σ is to be interpreted as a relation of indifference on the underlying set of worlds W : if w is connected by σ to v , this means intuitively that the difference between w and v is *not* at issue. Conversely, if w and v are not connected, the difference between them *is* at issue. I proceed now to the update semantics.

Definition 10 (Inquisitive semantics). The update of a state $\sigma \in \Sigma_P$ with a formula φ of a language L_P of InqL, in symbols $\sigma[\varphi]$, is inductively defined as follows:

$$\begin{aligned} \sigma[p] &= \{(i, j) \in \sigma : i(p) = j(p) = 1\} \\ \sigma[\perp] &= \emptyset \\ \sigma[\varphi \wedge \psi] &= \sigma[\varphi] \cap \sigma[\psi] \\ \sigma[\varphi \vee \psi] &= \sigma[\varphi] \cup \sigma[\psi] \\ \sigma[\varphi \rightarrow \psi] &= \{(i, j) \in \sigma : (\forall \iota \in \{i, j\}^2) \iota \in \sigma[\varphi] \implies \iota \in \sigma[\psi]\} \end{aligned} \quad +$$

I will shortly explicate Definition 10 by means of examples, but first, a few comments are in order. I will use the notation (i, j) to refer to the pair of worlds i and j in an inquisitive model as per Definition 9, as opposed to the more familiar notation $\langle i, j \rangle$. The reason for that is to make it more transparent that these are symmetric pairs, i.e., that the ordering in any pair of worlds in an inquisitive model σ is of no consequence, for, by definition of symmetry, $\langle i, j \rangle \in \sigma$ just in case $\langle j, i \rangle \in \sigma$.¹⁴ Consider in particular the definition of implication. It makes reference to the set

$$\{i, j\}^2 = \{\langle i, j \rangle, \langle j, i \rangle, \langle i, i \rangle, \langle j, j \rangle\},$$

or, in the notation I will adopt in this text, one of the following identical sets.

$$\begin{aligned} \{i, j\}^2 &= \{(i, j), (i, i), (j, j)\} \\ &= \{(j, i), (i, i), (j, j)\} \end{aligned}$$

The atomic and conjunction clauses are identical to the ones in Definition 5, and \perp is defined in the usual way for update semantics. Notice that I give separate clauses for each connective, this correctly suggests that the classical abbreviations defined for QL in Section 1.1.1 will *not* be valid in InqL . What formulas are and are not valid in InqL will be the topic of investigation of Chapter 2, so for the time being I will just alert the reader to the fact that several classical equivalences do not hold in InqL . To mention just a few notable ones,

$$\begin{aligned} \neg(\neg\varphi \wedge \neg\psi) &\rightarrow \varphi \vee \psi, \\ (\neg\varphi \rightarrow \neg\psi) &\rightarrow (\psi \rightarrow \varphi), \\ \neg\neg\varphi &\rightarrow \varphi, \end{aligned}$$

are all *invalid* in InqL .

Finally, let me remark that the very simple definition of disjunction above, purely by means of update potential union, embodies the expressive power of the system as an erotetic logic. As for implication, its apparent unwieldiness will be explicated later in this chapter and in Chapter 2.

InqL at work, a walk-through

Recall the picture of the ignorant and indifferent model on the language with only two atoms, repeated as Figure 1.4 on the next page.

Atoms in InqL behave just like in the partition system QL . For concreteness, consider we have an ignorant and indifferent state σ for the inquisitive language with two atoms: an update with p will keep only those pairs such that both elements make p true. This means that $\sigma[p]$ is as in Figure 1.5-a. An update of σ with q , then, gives us the model pictured in Figure 1.5-b.

Consider now $\sigma[p \vee q]$. By the disjunction clause, we get the union of the two states in Figure 1.5. This means that we disconnect from the model the world w_{00} , as one would expect, but that is not all the update with $p \vee q$ does. If we look at the representation of $\sigma[p \vee q]$, in Figure 1.6, we see that w_{01} and w_{10} are no longer connected to each other by σ .

¹⁴For this reason, the inquisitive models can be interpreted as undirected looping graphs, where the notation (a, b) for edges is standard.

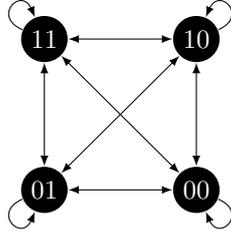


Figure 1.4: The indifferent, ignorant model σ for $P = \{p, q\}$

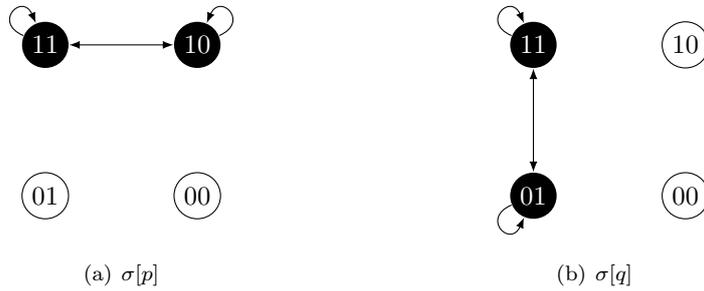


Figure 1.5: Two atomic updates

This means that the difference between worlds w_{01} and w_{10} interests us. Equivalently, the difference between these two worlds *is at issue*.

The partition system had the advantage of allowing us to move from the models construed as relations to the same models construed as sets of propositions via the partition theorem. For InqL , we must define another notion of what counts as an alternative in a model. These are highlighted in Figure 1.6.

Definition 11 (Alternatives). For σ a model of InqL , the set A_σ is the set of alternatives in σ , where $\alpha \in A_\sigma$ iff

1. $\alpha \subseteq \sigma$;
2. α is a total relation; and
3. there is no $\sigma \supseteq \beta \supset \alpha$ such that β is a total relation. +

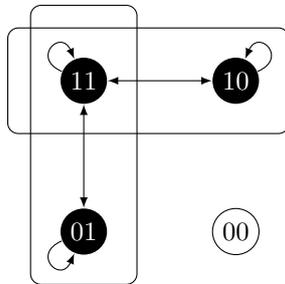


Figure 1.6: $\sigma[p \vee q]$

In other words, the set of alternatives in σ is the set of all maximally connected subsets of σ .

Since indifferent states are by definition total relations, we can equivalently define the set of alternatives for a state σ as the set of maximal indifferent substates of σ . In addition, we can also see that, for indifferent states, such as the ones in Figures 1.4 and 1.5, the set of alternatives is the singleton set containing the state itself. Thus, σ is indifferent just in case $A_\sigma = \{\sigma\}$. From this point onward, I will outline the alternatives in the pictures of inquisitive models, whenever the alternative set is different from the trivial singleton, that is, unless the state is indifferent.¹⁵

The state represented in Figure 1.6 however is not indifferent, for

$$A_{\sigma[p \vee q]} \neq \{\sigma[p \vee q]\} .$$

In fact, what we have is

$$A_{\sigma[p \vee q]} = \left\{ \begin{array}{l} \{(w_{11}, w_{11}), (w_{11}, w_{01}), (w_{01}, w_{11}), (w_{01}, w_{01})\}, \\ \{(w_{11}, w_{11}), (w_{11}, w_{10}), (w_{10}, w_{11}), (w_{10}, w_{10})\} \end{array} \right\} .$$

The two alternatives for $\sigma[p \vee q]$, outlined in Figure 1.6, correspond to two propositions, namely, p and q . Thus, the formula $p \vee q$, besides having provided the information that at least one of p and q must be the case (by eliminating the $\neg p, \neg q$ -world w_{00}), has also introduced an *issue* between the p -worlds and the q -worlds. That is, it has raised the issue of which one of p or q is the case. This is how inquisitive semantics encodes Grice's insight into natural language 'or'.

An update with a negated atomic formula has at least one unsurprising effect: $\neg p$ eliminates all the p -worlds from an information state. Interestingly though, the effect of double negation on disjunction is not vacuous as it was in the partition system QL . An update with $\neg(p \vee q)$ yields the single reflexive point (w_{00}, w_{00}) , and an update with $\neg\neg(p \vee q)$ gives us the complement of the latter information state.¹⁶ Figure 1.7 (next page) is a picture of $\sigma[\neg\neg(p \vee q)]$.

As one can see in Figure 1.7, the model $\sigma[\neg\neg(p \vee q)]$ embodies the same *information* as $\sigma[p \vee q]$. That is, if we abstract away from the relation and consider solely the subsets of W that $\sigma[\neg\neg(p \vee q)]$ and $\sigma[p \vee q]$ are relations on, we observe that they are identical: the same reflexive pairs have been eliminated from σ by the two updates. However, while the latter has two alternatives, the former has only one. Double negation has transformed an issue-raising disjunction into a purely informative, issueless update.

¹⁵This is of course not to say that indifference models lack alternatives! The alternative is simply a trivial one, for the only alternative is the model itself, and for that reason I will not tax the pictures with superfluous frames.

¹⁶Strictly speaking, it is inaccurate to see the update with a negation as the complement of the non-negated subformula, because problems would arise from such a definition when considering updates with conditionals or once-negated disjunctions. Consider the case $\neg(p \vee q)$, if this were to yield the complement of $\sigma[p \vee q]$ then, since the pair (w_{10}, w_{01}) is *not* a member of $\sigma[p \vee q]$, it would be present in $\sigma[\neg(p \vee q)]$. This would of course give rise to a non-standard model, for that information state would not be a reflexive relation (there would be a connection between two worlds whose reflexive pairs had been eliminated).

These problems prompted the slightly more complex definition I gave for the implication clause (recall that $\neg\varphi = \varphi \rightarrow \perp$). I will explain how this works in detail when I discuss the implication clause.

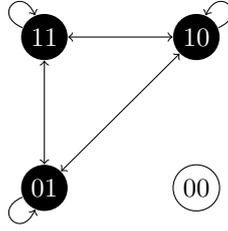


Figure 1.7: $\sigma[\neg\neg(p \vee q)]$

One way to interpret this issue-canceling power of negation is to interpret negated sentences as necessarily assertive, or perhaps more accurately, necessarily unquisitive. Then, double negation gets us the assertive closure of a certain formula, stripping away its inquisitive potential and leaving only its informative potential. Accordingly, I define an operator of assertive closure:

$$!\varphi := \neg\neg\varphi$$

Notice that I do not mean to imply that only sentences with ‘!’ are assertive in **InqL**. Rather, the idea is that all double negated sentences are necessarily assertive.

I move on to questions. As promised, **InqL** uses the abbreviation

$$?\varphi := \varphi \vee \neg\varphi$$

to formalize questions. It is easy to see how this gives **InqL** the full expressive power of the partition system QL . An update of σ with $?p$ is equivalent to an update with $p \vee \neg p$, which by the definition of disjunction means the union of the models for $\sigma[p]$ and $\sigma[\neg p]$. This is represented in Figure 1.8, with the two alternatives outlined.

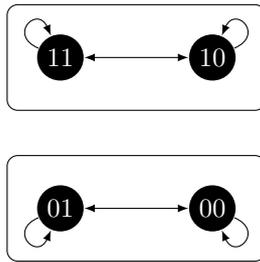


Figure 1.8: $\sigma[?p]$

Notice how an update with $p \vee \neg p$ is all but vacuous: clearly, $\sigma[p \vee \neg p]$, as represented in Figure 1.8, is a different state from the indifferent and ignorant σ as in Figure 1.4. Thus, $p \vee \neg p$ is *not* tautological, meaning that the law of excluded middle does not hold in **InqL**.

A few useful logical notions

So far I have been using terms such as ‘assertion’, ‘question’ and ‘informativeness’ in an intuitive, pre-theoretic way. What we have seen so far of how **lnqL** functions already allows us to be more explicit than that, as the following definitions demonstrate.

Definition 12 (Informativeness). A formula $\varphi \in L_P$ of inquisitive logic is *informative* iff, for ω_P the maximal model in Σ_P , $\omega_P[\varphi]$ is *not* ignorant (as per the straightforward adaptation of Definition 3 to inquisitive models). \dashv

Definition 13 (Inquisitiveness). A formula $\varphi \in L_P$ of inquisitive logic is *inquisitive* iff, for ω_P the maximal model in Σ_P , $\omega_P[\varphi]$ is *not* indifferent (as per the straightforward adaptation of Definition 3 to inquisitive models). \dashv

In other words, a formula is informative just in case it eliminates reflexive pairs, and it is inquisitive just in case it eliminates non-reflexive connections between worlds. These two notions allow us to define what we mean by ‘assertion’ and ‘question’ in a variety of ways, such as the following.¹⁷

Definition 14 (Assertions, questions and hybrids). A formula φ of any language of **lnqL** is an *assertion* iff it is not inquisitive and it is informative or tautological, a *question* iff it is inquisitive and not informative, and a *hybrid* iff it is both informative and inquisitive. \dashv

According to Definition 14, atomic formulas are assertions, for they are always purely informative, as are negated or double-negated formulas. Formulas prefixed with the question operator, or in other words formulas that are disjunctions of jointly inconsistent subformulas, are always questions, with the exception of the formula $?T$, which is equivalent to T and therefore an assertion.

Disjunctions of non-jointly inconsistent subformulas however, such as $p \vee q$, since they provide both information and issues (i.e., they are both informative and inquisitive), are *hybrids*. In addition, both contradictions and tautologies are assertions.¹⁸

Material implication in **lnqL** — a semantics for conditional questions

The free syntax of **lnqL** gives us, among others, $p \rightarrow ?q$ as a well-formed formula. This is the **lnqL** formalization of a conditional question, such as (22), with the possible answers in (23).¹⁹

¹⁷But see the literature on inquisitive semantics for different approaches and the motivations for them, in particular, Groenendijk (2008b) and Groenendijk and Roelofsen (2009).

¹⁸It is perhaps easy to see how a contradiction should be an assertion, it is the absurd assertion. As for the tautology, this is a purely technical move, motivated by the idea that the assertive operator ‘!’ ought to characterize the class of assertions: the tautology, just like any other uncontroversial assertion, is unaffected by ‘!’. Nevertheless, it is also possible to stipulate that the tautology is a question.

¹⁹Interrogative conditionals are I believe ambiguous and can get one of two readings, one where the interrogation operator takes narrow scope over the consequent, and one where it seems to take wide scope, over the whole conditional. Although the situation with interrogation might be quite close to that of negation, inasmuch as narrow scope is probably the preferred reading for both negated and interrogative conditionals, I find that the wide scope

- (22) If Jake goes to the party, will Mary also go?
 $p \rightarrow ?q$
- (23) a. Yes. (If Jake goes, then so will Mary.)
 $p \rightarrow q$
 b. No. (If Jake goes, then Mary won't go.)
 $p \rightarrow \neg q$

An update with $p \rightarrow ?q$ on the minimal information state I have been using for expository purposes in this section yields the state depicted in Figure 1.9. As

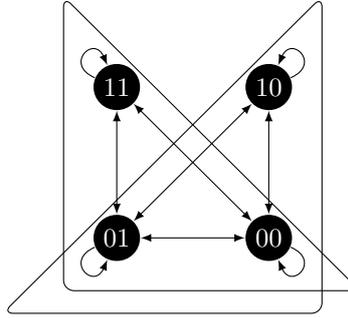


Figure 1.9: $\sigma[p \rightarrow ?q]$

the picture shows, the update is uninformative in our technical sense; rather, it creates an issue with two alternatives that correspond to the propositions $p \rightarrow q$ and $p \rightarrow \neg q$, as discussed in Section 1.1.2.

It is important to note that, in InqL , a formula of the sort $\varphi \rightarrow ?\psi$, whenever φ is an assertion, is equivalent to a disjunction of non-inquisitive implications. Thus, for φ an assertion,

$$\varphi \rightarrow ?\psi \iff (\varphi \rightarrow \psi) \vee (\varphi \rightarrow \neg\psi) ,$$

or indeed, in general,²⁰

$$\neg\varphi \rightarrow \psi \vee \theta \iff (\neg\varphi \rightarrow \psi) \vee (\neg\varphi \rightarrow \theta) .$$

reading is often available, and sometimes quite prominent. A good way to discern between the two readings is to check the meaning of a negative answer to an interrogative conditional. Compare (22) and the meaning of its negative answer to the example below:

- (i) If I learn to play the violin, will I get a job at the BSO?
 No (it's possible that you'll learn the violin and still not get the job at the BSO).

If the negative answer to (i) means the assertion between parentheses, then the question was not of the type in (22), but rather something that might be formalized, in a modal InqL , as $? \Box(p \rightarrow q)$, a question about the necessity of a material implication, roughly, a question about the (possibly causal) relation between the two events in the conditional. Such a formula would generate the alternatives $\Box(p \rightarrow q)$ and $\Diamond(p \wedge \neg q)$ (for this is equivalent to $\neg \Box(p \rightarrow q)$), as intended. I will disregard this reading of interrogative conditionals, on the grounds that it can only be dealt with in an inquisitive modal logic, which is not the object of discussion in this thesis. In any event, it is I believe clear that $?(p \rightarrow q)$ cannot be the correct way to model this meaning, given its answerhood conditions as discussed in the main text.

²⁰For recall that any negated formula is an assertion.

The latter formula is known as the **KP** axiom, or, as an admissible rule of intuitionistic logic, the principle of *Independence of Premise*. I will return to it in Chapter 2.

The inner workings of the implication clause are somewhat tedious to show, but important for a thorough understanding of InqL .²¹ I will first show how the update with $p \rightarrow ?q$ yields the desired result, and I will conclude this section with a brief discussion of negation (for recall that negation is defined in terms of implication and \perp .)

The definition of the update of a state σ with $p \rightarrow ?q$ unfolds into the following set.

$$\left\{ \begin{array}{l} (i, j) \in \sigma : (i, j) \in \sigma[p] \implies (i, j) \in \sigma[q \vee \neg q] \quad \& \\ (i, i) \in \sigma[p] \implies (i, i) \in \sigma[q \vee \neg q] \quad \& \\ (j, j) \in \sigma[p] \implies (j, j) \in \sigma[q \vee \neg q] \end{array} \right\}$$

Now, considering, for concreteness, our usual ignorant and indifferent σ for the language with two propositional atoms, it is easy to see how pairs involving the worlds where p is *false* will be part of this set: such pairs, by the atomic clause, will always *fail* an update with p , thereby trivially satisfying the conditionals above. $\sigma[p \rightarrow ?q]$ will therefore certainly contain the following pairs.²²

$$\begin{aligned} &(w_{01}, w_{01}), (w_{00}, w_{00}), \\ &(w_{01}, w_{00}), \\ &(w_{01}, w_{11}), (w_{01}, w_{10}), \\ &(w_{00}, w_{11}), (w_{00}, w_{10}) \end{aligned}$$

But the interesting pairs are the ones that *do* survive an update with p , therefore not trivially satisfying the conditionals. These are

$$(w_{11}, w_{11}), (w_{10}, w_{10}), (w_{11}, w_{10}) .$$

The reflexive pairs are in $\sigma[p \rightarrow ?q]$, for they satisfy the consequent of the conditionals above, in that they survive a further update with $q \vee \neg q$. The reason they do is that reflexive pairs are essentially total valuations, they are just like fully specified possible worlds, and as such undoubtedly satisfy excluded middle.

The same is not true of the pair (w_{11}, w_{10}) above. For this pair to be in $\sigma[p \rightarrow ?q]$, it would have to be the case that 1. the reflexive pairs built out of worlds w_{11} and w_{10} survive an update with $q \vee \neg q$ and 2. the pair (w_{11}, w_{10}) itself survives an update with $q \vee \neg q$. 1. as we have seen holds, but not 2. Indeed, for (w_{11}, w_{10}) to survive an update with $q \vee \neg q$, by definition of disjunction, it would have to survive an update with either q or $\neg q$. Clearly, it survives neither of the two, for q is true at w_{11} and false at w_{10} . Therefore, the pair (w_{11}, w_{10}) is eliminated from σ by the update with $p \rightarrow ?q$.

Also in connection with implication in InqL , it is important to explicate how (and why) negation is defined in terms of it and \perp . The update of a state σ

²¹Readers less interested in gory logical details can safely skip the next few paragraphs, if they are willing to grant me that the implication clause works as intended.

²²Keep in mind that, according to the notation I use here, (i, j) means $\langle i, j \rangle$ and $\langle j, i \rangle$.

with $\neg\varphi$ (i.e., $\varphi \rightarrow \perp$) is, by definition

$$\left\{ \begin{array}{l} (i, j) \in \sigma : (i, j) \in \sigma[\varphi] \implies (i, j) \in \sigma[\perp] \quad \& \\ (i, i) \in \sigma[\varphi] \implies (i, i) \in \sigma[\perp] \quad \& \\ (j, j) \in \sigma[\varphi] \implies (j, j) \in \sigma[\perp] \end{array} \right\},$$

which, since $\sigma[\perp] = \emptyset$, is equal to

$$\left\{ \begin{array}{l} (i, j) \in \sigma : (i, j) \notin \sigma[\varphi] \& \\ (i, i) \notin \sigma[\varphi] \& \\ (j, j) \notin \sigma[\varphi] \end{array} \right\}.$$

In words, a pair (i, j) survives an update with $\neg\varphi$ just in case it and the two reflexive pairs that underlie it *fail* an update with φ . This provides a good example of why we need the complicated implication clause defined above.

If we were to use a more straightforward definition, such as

$$\sigma[\varphi \rightarrow \psi] = \{(i, j) \in \sigma : (i, j) \in \sigma[\varphi] \implies (i, j) \in \sigma[\psi]\},$$

then negation would amount to simply the following, which is complementation.

$$\sigma[\neg\varphi] = \{(i, j) \in \sigma : (i, j) \notin \sigma[\varphi]\}$$

Now, recall that $\neg(p \vee q)$ eliminated from σ the pair (w_{10}, w_{01}) . Since this pair is not in the update with $p \vee q$, it must be a part of the update with $\neg(p \vee q)$, which would yield a non-reflexive model; for, obviously, (w_{10}, w_{10}) would not be a part of that model, since it survives the update with $p \vee q$. This non-standard model is unacceptable (and uninterpretable under Hulstijn's heuristic), so this would be a catastrophic result.²³

1.2.3 More applications of InqL

So far I have defined the system **InqL** and shown its inner workings by means of several examples. I have already shown that **InqL** addresses issues with the partition approach raised in Section 1.1.2. In particular, **InqL** has a free syntax that allows for inquisitive and informative sentences to be combined in any way, while attributing them a meaning. Moreover, by dropping the partition requirement, **InqL** can express a whole new class of meanings, such as conditional questions and inquisitive disjunctions. In the following sections I will discuss further characteristics of **InqL**.

Disjunctions meet questions

At the crux of the expressive power of **InqL** lies, as I have stressed before, the issue-raising potential of disjunction. However, simple disjunction, as in $p \vee q$, has a close, non-inquisitive cousin, namely $!(p \vee q)$, shorthand for $\neg\neg(p \vee q)$. Do these two flavors of disjunction have linguistic correlates? I am convinced that they do, and I will provide two lines of argument for that.

²³There are further examples of the need for the implication clause as I have defined it that do not involve negation, but I omit them in the interest of brevity. The interested reader can try the exotic formula $?p \rightarrow ?q$, first with the straw man simple implication defined above, to see what the problem with it is, and then with the clause in Definition 10, to see how that solves it.

Given the heuristic I have proposed for the models of InqL , the difference between $p \vee q$ and $!(p \vee q)$ is minimal: they are informatively equivalent and differ only in that the former raises an issue regarding which one of p or q happens to be the case, whereas the latter introduces no issue whatsoever. Consider the following two scenarios and sentences of German.

Scenario 1

You and I are hosting a party this evening and have invited John and Mary.

- (24) John oder Mary kommt heute zur Party.
 John or Mary come-**sg** today to-the party
 ‘John or Mary is coming to the party tonight.’
 $p \vee q$

Scenario 2

You and I are newlyweds, and are having a party this evening to celebrate our return from our honeymoon. John and Mary are a couple who were invited to our wedding but have so far failed to give us a wedding gift. Chances are, they won’t have the nerve to appear at our party this evening without finally bringing the gift. Luckily, I have just gotten a message from John saying that either he, or Mary, or both will come to the party this evening.

- (25) John oder Mary kommen heute zur Party.
 John or Mary come-**pl** today to-the party
 ‘John or Mary is coming to the party tonight.’
 $!(p \vee q)$

A majority of the German speakers I presented these scenarios and sentences with found (24) more felicitous with Scenario 1 than with 2 and conversely for (25).²⁴ I believe inquisitive semantics can shed some light on these facts.

In (24), we have a pure, hybrid disjunction of the form $p \vee q$, which raises the issue of which one of John or Mary is coming to the party. Given Scenario 1, it is felicitous to utter (24) and thereby draw attention to our ignorance about who exactly is coming to the party, as well as express interest in finding out. (24) is infelicitous under Scenario 2 because there it is quite evident that we care little about John or Mary, or which one of the two, if not both, is coming to the party. We are happy to know that at least one of them will come (and bring our wedding gift), and it would be infelicitous to draw attention to our ignorance

²⁴Jeroen van Craenenbroek (p.c.) tells me that the same contrast holds in Dutch, and it is present as well in (at least European) Portuguese. My unreliable intuitions about English hint at the existence of the same contrast, but this may well be an influence of Portuguese. Sadly, it is very difficult to get speakers of English to admit that they ever use plural agreement with ‘or’-conjoined noun phrases, so my inquiries were more often than not met with a sound “‘John or Mary are coming’ is absolutely bad in any context!” In actuality, both singular and plural agreement are possible in English, as a simple Google search for the string “or Mary are” can attest. Here’s an example, with thanks to Anna Szabolcsi.

- (i) When we phone for any advice or want additional equipment *Seamus or Mary are* always on hand to deal with us quickly and professionally.
<http://www.telecomservices-ire.com/testimonials.php>

or to express interest in who exactly will bring the gift. In this setting, a purely informative sentence such as (25), with an assertively-closed disjunction, is the only felicitous utterance of the two.

The fact that speakers show a preference for one German sentence over the other based on number agreement with the subject on the verb suggests that this feature might determine which flavor of disjunction (plain or assertive-closed) to use in interpreting the sentence, and provides some linguistic backing to InqL’s prediction that disjunctions comes in two kinds.

A more well-known example of these two flavors of disjunction can be seen in the following:

- (26) A: Do you want **coffee** or **tea**?
 B: Coffee, please.
 B: Tea, please.
 B: Neither(, actually).
 B: % Yes.
 B: % Yes, coffee.
- (27) A: Do you want **coffee** or **tea**?
 B: Yes / Sure.
 B: (%) Yes, coffee.
 B: % Coffee, please.

In (26) and (27), A is asking two different questions (a fact reflected in the intonation patterns): (26) is an alternative question, and (27) a polar, *yes-no* question. Notice that, when B answers A’s question in (27) by saying “Yes, coffee”, B is giving more information than required. In particular, a *yes* answer is required first, as the contrast with “Coffee, please” shows (without B nodding or otherwise explicitly indicating a simple affirmative answer).²⁵

Consider now the following two InqL formulas and the alternative sets they generate:

- (28) $?(p \vee q)$
 Alternative set = $\{p, q, \neg(p \vee q)\}$
- (29) $?!(p \vee q)$
 Alternative set = $\{!(p \vee q), \neg(p \vee q)\}$

The two updates are questions, in the sense that they are purely inquisitive, and they seem to correspond quite intuitively to the natural language questions in (26) and (27), respectively. In particular, they generate the appropriate alternative sets.²⁶

The examples in (28) and (29) further illustrate how the alternatives generated by a simple disjunction contribute to the meaning of an alternative question like the one in (26). The three possible answers could only be generated because of the inquisitive quality present in simple disjunction. If we override

²⁵These are, as everyone knows, idealized judgments. In practice, the oddness markers in front of B’s responses in (27) are extremely subtle, given how easy it is to accommodate a tacit affirmative answer when B’s reaction is to specify the beverage she wants.

²⁶It is a much debated issue whether the *neither* answer to the question in (26) has the same status as the *coffee* and *tea* answers. Many people find it unnatural without ‘actually’, which might indicate that it is not a complete answer *per se*, in that it rejects a presupposition. I will not pursue the matter here.

disjunctive inquisitiveness, as in (29), by double-negating a disjunction, we get the often unnatural, ‘logician’s’ interpretation of a disjunction (“Shall we go to the movies or the theater? Yes.”).^{27,28}

The last instance of interaction between questions and disjunctions I will discuss is the case of disjunctions of questions in lnqL , that is, formulas of the sort

$$?\varphi \vee ?\psi .$$

These, I argue, are good candidates for *choice questions*.

I gave an example of a choice question discourse in Section 1.1.2, involving a question in a set-theory exam. There are however natural occurrences of interrogative sentences that can be analyzed as choice questions. Suppose you see a child at an amusement park looking distressed. Assuming the child is lost, you might ask him the following question.

(30) Where is your father or your mother?

This question offers the child a choice between telling you where his mother is and where his father is. You will be satisfied with an answer to either question.

Now, it is probably impossible to find natural examples of polar choice questions where ‘or’ operates at the sentential level, as in (31).

(31) ??Is Mary home? Or is the party over?

Szabolcsi (1997) argues that matrix disjunctions of questions in general are absent from natural languages. In English, for example, sentences like “Who did you marry or where do you live?” where *or* is meant to operate at the inter-sentential level, are also extremely odd, to say the least. Moreover, Hungarian requires that one add a word to the effect of ‘rather’ or ‘instead’, suggesting that matrix disjunctions of questions are really not choice questions but rather “an idiomatic device that allows one to cancel the first question and replace it with the second” (*ibid*). Be that as it may, at least (30) is an example where disjunction, though perhaps applying at the NP level, interacts with interrogation to yield a grammatical question with a choice function interpretation.

lnqL as investigated in this thesis is a propositional system, so it cannot exactly formulate the question in (30), which I’m analyzing as a disjunction of two constituent questions *where*. However, under certain assumptions about the domain and our information state, (30) can in fact be formulated as a disjunction of questions isomorphic to polar questions, as follows.

Let Mom and Dad be the names of the only two individuals in the domain, and let Here and There be the names of the only two locations in the domain;

²⁷I have recently learned that the joke works the other way around too. I hope the reader familiar with the original anecdote will forgive me, since I cannot recall who exactly its protagonist was, but there is the story of an important intellectual being interrogated during the McCarthy era: when asked “do you believe that the American people ought to be allowed to advocate overthrowing the US government by force or other violent means?”, the person under interrogation thought for a while, and then replied “By force.”

²⁸Notice that, as Han and Romero (2001) point out, simple negation, when it takes wide scope over disjunction, also bars the alternative question interpretation:

(i) Don’t you want coffee or tea?

Clearly, this is a *yes-no* question, as is indeed predicted by lnqL , for $?\neg(p \vee q)$ has the same possible answers as $?!(p \vee q)$, and not those of $?(p \vee q)$.

further assume that neither Mom or Dad can be in two locations at the same time and that each must be in one location at any given moment. Consider now Figure 1.10, where we interpret the first digit in each world to give the truth value of the proposition “Mom is Here” and the second digit that of the proposition “Dad is Here.”

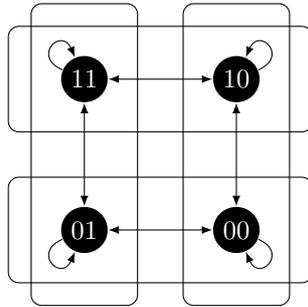


Figure 1.10: $\sigma[?p \vee ?q]$

Each world represents one of the four possible states of affairs. w_{11} is the world where Mom and Dad are Here; w_{10} the world where Mom is Here and Dad isn’t Here, which entails that Dad is There; and so on. Under this interpretation, Figure 1.10 gives exactly the kind of meaning we want for (30): the four alternatives available correspond to the four different propositions of the form “ x is in l ,” so any answer of that form will bring us to a state of indifference, addressing the issue raised.

Figure 1.10 can also be interpreted as a representation of the update of σ with the formula $?p \vee ?q$, which unfolds into $(p \vee \neg p) \vee (q \vee \neg q)$, essentially a list of the propositions that the addressee of the question is to choose from. Similarly, the question in (30), under the assumptions made above, gives a list of answers that would satisfy the questioner, (semi-) formally, $(\text{Here}(\text{Mom}) \vee \text{There}(\text{Mom})) \vee (\text{Here}(\text{Dad}) \vee \text{There}(\text{Dad}))$. Therefore, under these assumptions, $?p \vee ?q$ is isomorphic to the meaning of (30). The significance of this result is the following: although it may well be the case that there are no matrix disjunctions of polar questions in natural languages, there are constituent questions that at least under certain assumptions correspond to meanings isomorphic to those of disjunctions of propositional questions in InqL . The fact that those assumptions were extremely strong is no reason for concern: the sole purpose of this exercise was to show that non-partitioned meanings of the sort generated by formulas like $?p \vee ?q$ can be useful to model certain natural language meanings.

Moreover, the exam question discussed earlier in this chapter, although it is an inquisitive discourse rather than an interrogative sentence, can be modeled in terms of a disjunction of polar questions.

- (32) Answer one of the following two questions.
- a. Is the collection of all infinite sets a set?
 - b. Is the Axiom of Choice equivalent to the Well-Ordering Theorem in ZF?

If we take (32-a) to be $?p$ and (32-b) to be $?q$, then (32) can be formalized as $?p \vee ?q$, giving rise to that state depicted in Figure 1.10.

Alternative semantics for natural language ‘or’

The alternative-generating potential we see in disjunctions can be seen as an implementation of so-called Hamblin semantics for disjunction (see for example Alonso-Ovalle, 2006), as I mentioned in Section 1.2.1. The intuitive idea behind these alternative semantics is that natural language ‘or’ is different from the simple Boolean join of two propositions, rather, it yields a set of alternative propositions. This idea has proven fruitful in dealing with the free choice problem, the exclusiveness implicature and other non-standard inferences legitimized by natural language disjunction.

InqL is an alternative semantics in this sense as well, in that the essential premise of Hamblin semantics, namely, that disjuncts be accessible to higher operators, is to some extent implemented in InqL.²⁹

A classical illustration of the use of alternative semantics for disjunctions comes from the interaction between modality and disjunction. It is well known that sentences like (33-a) are preferably interpreted as in (33-b).

- (33) a. John may be in Paris or in London.
b. John may be in Paris and he may be in London.

Schematically, this corresponds to an inference of the sort

$$\diamond(\varphi \vee \psi) \vDash \diamond\varphi \wedge \diamond\psi ,$$

which of course is invalid in traditional epistemic frames. Alternative semantics account for this inference (reproducible in other modal domains besides the epistemic one) by postulating that the modal can distribute over the disjuncts. However, if disjunction is Boolean join, then the modal has only access to a “blob” of possible worlds, the union of the propositions φ and ψ , and cannot distribute. Enter alternative-generating disjunctions: if ‘or’ is *not* the Boolean join, but rather an operator that generates a set with two propositions, *phi* and *psi*, then the modal can easily access the individual disjuncts and distribute over them.

Clearly, InqL has the means necessary to offer an equivalent account of free choice, and presumably of related phenomena. Indeed, the similarity between inquisitive and alternative semantics is quite striking: both semantics take the Hamblin route of generalizing “set semantics,” that is, both semantics take even atomic sentences to mean not just propositions but rather proposition sets, and both semantics assume that disjunction can generate alternatives.

²⁹InqL has a weaker expressive power than that which is usually assumed for Hamblin semantics. For example, InqL generates disjunctive alternatives up to logical entailment, which is not a restriction implemented upon any of the Hamblin semantics I am familiar with. Consider the sentence below:

- (i) Phillip lives in Massachusetts or Boston.

The alternative semantics I am referring to would assign two alternatives to (i), while in InqL we would have only one, namely the weaker “Phillip lives in Massachusetts”, for the other proposition asymmetrically entails this one. This is a consequence of the InqL system which, at least in this case, seems to be an advantage over other alternative semantics for disjunction, insofar as the sentence in (i) is not a felicitous use of NL disjunction — InqL correctly prevents it from generating the alternatives it intends to. In the conclusion to this chapter I will discuss another instance of InqL’s expressive weakness.

They come apart however in at least three respects. First, inquisitive semantics has a logic, **lnqL**, with fully worked-out properties, which alternative semantics lacks. Second, inquisitive semantics has a broader program of bringing interrogation and disjunction together, which alternative semantics, to the best of my knowledge, has not considered. And third, the expressive powers of both systems are different.³⁰

At any rate, **lnqL** has the tools necessary to account for at least a portion of the free-choice and related data other alternative semantics on the market cover, while making use of the same basic insight and enriching it, by establishing the connection with interrogation.

Overgeneration?

Its free syntax is what allows **lnqL** to be such a simple and elegant erotetic logic, and yet powerful enough to account for non-partitioning questions, issue-raising disjunctions, and so on. However, this free syntax also gives rise to such oddities as the following, which I introduce roughly in ascending order of awkwardness:

- | | | |
|-------------|-----------------|------------------------|
| 1. $\neg?p$ | 3. $??p$ | 5. $?p \rightarrow q$ |
| 2. $!?p$ | 4. $?p \vee ?q$ | 6. $?p \rightarrow ?q$ |

Naturally, the odd quality of these formulas only arises under the interpretation of $\varphi \vee \neg\varphi$ as a question. That interpretation, although it is not a primitive in our semantics, is of course essential to establish the viability of **lnqL** as an erotetic logic, so it cannot be disregarded at will.

The first three formulas are instances of negation of questions and of iteration of the question operator. Now, since the question operator is not a primitive, these boil down to, respectively, $\neg(p \vee \neg p)$, $\neg\neg(p \vee \neg p)$ and $p \vee \neg p$. Thus, the negation of an atomic question is equivalent to \perp , the double negation of an atomic question to \top , and the iteration of questions has no effect. Notice that, while these formulas all get a semantic interpretation, their syntax is still odd and natural language correlates to them seem impossible to find. The negation of a question is completely absent from natural language, and, granted, assigning to it the interpretation of the contradiction may be considered acceptable. But how is the assertion of a question a tautology? Doesn't that sound just as "contradictory" as the simple negation of a question?

Again, my response must be that 1. this free syntax has clear formal advantages and 2. **lnqL** is designed to be a literal and radical instantiation of the inquisitive program. Certainly, we must constrain the syntax of a logic for natural language questions to the same extent that natural language syntax is constrained.

Ideally, of course, we won't have to purely stipulate constraints and will be able to reduce them to broader principles. Natural language syntax / Logical-Form semantics, for example, have an explanation of why negation cannot take scope over interrogation, in terms of *wh*-movement to a locus higher than the site hosting negation. For that reason, syntactically minded semantics of questions can block such inconvenient formulas as the ones above more or less easily. There is therefore no reason not to conjecture that the next, more linguistically adequate instantiation of inquisitive semantics will be able to use the same

³⁰See footnote 29.

mechanisms to block the undesired formulas, while still being able to assign meanings to such formulas, were they ever to occur. In short, the capacity of **lnqL** to interpret (linguistically) bizarre formulas need not be seen as a flaw; it is merely a sign of the fact that there is work to be done in designing an inquisitive semantics that is based on **lnqL** and yet is linguistically more adequate.

1.2.4 Final remarks

I began this chapter by casting doubt on the influential theory that question meanings correspond to partitions of the logical space. I attacked the partition theory on three fronts:

1. It sharply distinguishes informativeness and inquisitiveness, assertions and questions;
2. partitions strongly restrict the class of meanings that can be expressed by a language, while there are good reasons to believe that natural languages can express (sometimes via simple sentences, other via more elaborate discourses) question meanings that warrant a greater expressive power;
3. partitions are not a conceptually necessary postulate, contrary to what some semanticists and philosophers have assumed; to the contrary, there is a compelling conceptual argument against partitionhood.

I then introduced the inquisitive system **lnqL**, which addresses all of the above issues while at the same time incorporating a well-known linguistic insight, namely that natural language questions and disjunctions are morphologically related in many natural languages. The main postulates of inquisitive semantics are repeated below.

- (34)
- a. Semantic alternatives are generated by the mechanism of *raising issues*.
 - b. Disjunction is the primitive issue-raising operator
 - c. **Corollary:** Questions are a special case of disjunctions.

Postulate (34-a) is motivated by the fact that alternative-generating operators in language, in particular questions and disjunctions, convey ignorance and raise issues, and (34-b) is motivated by the fact that, in languages where the disjunction operator and the question operator are morphologically related *but distinct*, the latter is derived from the former, and not the other way around. The corollary in (34-c) is a consequence of the postulates, when taken quite literally, and it is what motivates the abbreviation

$$?\varphi := \varphi \vee \neg\varphi .$$

These three premises are behind the inquisitive system **lnqL**, a simple propositional logic with a standard language, interpreted in independently motivated models, that nevertheless addresses the problems with the partition theory raised above.

There are a number of loose ends in this chapter. One I find especially intriguing is the connection between this work and indefinites, alluded to in footnote

10. Indefinites too have been argued to generate alternatives (Kratzer and Shimoyama, 2002), which suggests that they should be integrated in an inquisitive program. Moreover, a subset of the natural languages that show morphological relatedness between questions and disjunctions also have indefinites joining the unholy mix, so the linguistic motivations for identifying questions and disjunctions can be made with respect to disjunctions and indefinites.

The connection between indefinites and disjunction, since it must be spelled out in a first order inquisitive logic (Groenendijk, 2008c), lies outside the scope of this thesis. I do however want to remark that, if indefinites are to be identified with (a generalized quantifier employing) the existential quantifier of first order logic, then a first order inquisitive semantics has a straightforward mechanism to generate alternatives for indefinites in terms of disjunction in the meta-level. Without defining models or a complete semantics, let me just give an informal update semantics clause for the existential quantifier that illustrates this point. Let A be the domain of quantification (the fixed domain in the model or perhaps a contextually restricted one):

$$\sigma[\exists x.\varphi(x)] = \bigcup_{a \in A} \sigma[\varphi(a)]$$

That is, an update with $\exists x.\varphi(x)$ equals the union of the individual updates with φa for each a in the domain. It is easy to see how this generates the intended alternatives via disjunction, i.e. set-union at the meta-level.

To the extent that indefinites can be taken to be generalized quantifiers of this sort, it is an encouraging property of first order inquisitive logic that its existentially quantified sentences can raise alternatives in much the same way as its disjunctions.^{31,32}

Standard erotetic logics typically use primitively defined question operators. This gives rise to a flat, one-dimensional view of questions and assertions; for, in those approaches, the building blocks of formulas are exclusively informative and exclusively inquisitive subformulas. Data and issues are kept separate at the level of semantic definitions, only arising in some of those logics in the form of hybrid formulas that are restricted by the syntax.

In **lnqL**, however, we achieve the intended expressive power by choosing the hybrid quality of natural language disjunction as our starting point and defining questions in terms of it. This gives us a two-dimensional view on informativeness and inquisitiveness. The dichotomy between these two views is represented in Figure 1.11.

This two-dimensional perspective on the informative and inquisitive power of **lnqL** formulas is embodied in the fact that there is no primitive clear distinction, syntactic or semantic, between assertions and questions. Naturally, a distinction

³¹It is a separate matter whether we can bring this similarity between indefinites and disjunctions into the object language.

³²Note that this first order logic predicts that, just like with disjunctions, indefinites come in two flavors, one “plain” the other double-negated and therefore non-inquisitive. It would be interesting if we could find a parallel between the hybrid and purely informative uses of disjunction discussed earlier in this chapter and two different uses of indefinites in natural language. Jeroen Groenendijk has suggested that the inquisitive version of existential formulas can be interpreted as giving rise to a specific reading of an indefinite, while the purely informative variety would correspond to a non-specific or quantificational use of an indefinite.

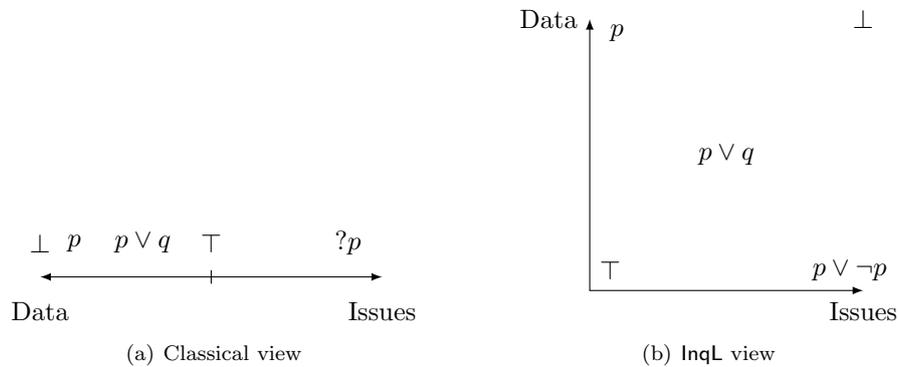


Figure 1.11: Two views on data and issues

can be defined over the syntactic and semantic primitives I gave; that will be the likely future of inquisitive logic as it is applied to natural language semantics.

The idea that the meaning of a natural language disjunction isn't merely propositional is by no means new to inquisitive semantics. In so-called alternative semantics for disjunction, the disjunction operator is not identified with the Boolean join, which produces a simple union of propositions; rather, it takes a number of propositions and yields a set of propositions, much like the interpretation InqL assigns to a formula like $p \vee q$. In a way then, InqL can be seen as a logic that brings together insights from the semantics of questions and the semantics of disjunction and offers a unified view of the mechanism for generating alternatives, while addressing problems with the partition theory of questions.

Chapter 2

Inquisitive Logic

This chapter presents the results of my investigations on the logical properties of the system defined in the previous chapter. I will begin with some informal remarks and then proceed to introduce two crucial properties of the inquisitive system that have philosophical and linguistic relevance. I will then introduce an alternative formulation of the semantics of inquisitive logic in classical terms (that is, no update functions), which I will use to present a sound and complete axiomatization of the system.

2.1 InqL as an update logic

2.1.1 Preliminary remarks

Readers familiar with update logics have no doubt already asked themselves while reading the first chapter (and most likely have found an answer to) whether the inquisitive logic there defined has the properties of eliminativity and distributivity. A simple look at the recursive definition of the update functions, repeated below, is enough to get an answer.

Definition 15 (Inquisitive semantics). The update of a state $\sigma \in \Sigma_P$ with a formula φ of a language L_P of InqL is inductively defined as follows:

$$\begin{aligned}\sigma[p] &= \{(i, j) \in \sigma : i(p) = j(p) = 1\} \\ \sigma[\perp] &= \emptyset \\ \sigma[\varphi \wedge \psi] &= \sigma[\varphi] \cap \sigma[\psi] \\ \sigma[\varphi \vee \psi] &= \sigma[\varphi] \cup \sigma[\psi] \\ \sigma[\varphi \rightarrow \psi] &= \{(i, j) \in \sigma : (\forall \iota \in \{i, j\}^2) \iota \in \sigma[\varphi] \implies \iota \in \sigma[\varphi][\psi]\} \quad \dashv\end{aligned}$$

One can immediately see that the system is eliminative, that is, for any formula φ in the language, for any state σ , $\sigma[\varphi] \subseteq \sigma$. An informal inductive argument should be enough: the atomic and implication clauses clearly select a subset of the original σ ; conjunction and disjunction, given the induction hypothesis, are the intersection, respectively, union of two subsets of σ , and thus themselves necessarily subsets of σ .

An update system is distributive if, for all formulas φ and states σ , the result of updating σ with φ is equal to the union of the updates of the *points* in σ with

φ , where the set of points in σ is the set of all singletons from elements of σ . Again, an informal argument suffices: notice that the implication clause has a search space that is larger than the point of evaluation, namely, to check whether a pair (i, j) models an implication, we must check whether the reflexive pairs thereof model the implication. Thus, **lnqL** as an update logic is *not* distributive.

One final remark is in order, in this preliminary discussion. As one can see by the fact that $\varphi \vee \neg\varphi$ is quite crucially *not* a tautology of the logic, **lnqL** is less than classical. In fact,

$$\mathbf{CPL} \subset \mathbf{lnqL} \subset \mathbf{IPL} ,$$

that is, **lnqL** is an intermediate logic, properly included in **CPL** and properly including intuitionistic propositional logic. That **lnqL** \neq **IPL** is easy to show, by simply pointing out a valid formula of **lnqL** that is intuitionistically invalid; one such example is the formula that axiomatizes the Kreisel-Putnam logic **KP**, namely

$$(\neg\varphi \rightarrow \psi \vee \theta) \rightarrow (\neg\varphi \rightarrow \psi) \vee (\neg\varphi \rightarrow \theta) ,$$

which I mentioned in the previous chapter, while discussing conditional questions. A semantic proof of this validity can be produced, but its length far surpasses its interest, so I'll omit it. As we will see in later sections, it is no surprise that the **KP** axiom is valid in **lnqL**, as **lnqL** happens to be an extremely strong logic. I will discuss the axiomatization of **lnqL** in later sections, for now it will suffice to keep in mind that **lnqL** is less than classical and more than intuitionistic¹.

2.1.2 Functional completeness

A static logical system is functionally complete if the language is able to express all truth functions. While this notion clearly cannot be directly imported into update logics, as the latter systems do not deal with truth functions, we can offer an update version of the concept of functional completeness that keeps the basic intuition behind the original meaning. Definition 16 below states that a language of inquisitive semantics is functionally complete just in case, from any given state in the corresponding class of models we can move to any smaller state via an update with a sequence of formulas.

Definition 16 (Functional Completeness). An update logic L_P , for P a finite set of propositional atoms, is functionally complete iff, for all $\sigma, \sigma' \in \Sigma_P$ such that $\sigma' \subseteq \sigma$, there is a finite sequence of updates $\tau = \langle \varphi_0, \varphi_1, \dots, \varphi_n \rangle$, such that $\sigma[\tau] = \sigma'$. ⊣

Notice that I am using sequencing in Definition 16, instead of the more standard conjunction. The reason for that is that we want to prove functional completeness for a minimal set of connectives, namely $\{\vee, \neg\}$, and dispensing with conjunction in the definition makes this possible. One can argue that using sequencing to do the job of conjunction is misleading, for sequencing is *not* part of the inquisitive languages discussed so far. I'm sympathetic to that criticism,

¹Strictly speaking, I have not shown that inquisitive logic includes intuitionistic logic. The results I will present later on the axiomatization and completeness of **lnqL** will make this quite clear in a simple way.

and suggest that the dissatisfied reader take Theorem 20 to show that the set of connectives $\{\vee, \neg, \wedge\}$ is functionally complete, as this involves a minor and obvious adaptation of the proof.

Before going into the relevant functional completeness theorem for InqL , I will present a vital definition, that of characteristic formulas.

Definition 17 (Characteristic Formulas). The *characteristic formula* for a possible world $i \in W$ is the formula χ_i defined as follows:

$$\chi_i := \neg \bigvee \{ \{ \neg p_n : p_n \in P \text{ and } i(p_n) = 1 \} \cup \{ p_n : p_n \in P \text{ and } i(p_n) = 0 \} \} \quad \dashv$$

Characteristic formulas single out specific worlds, as they are modeled at the worlds they characterize and nowhere else. Accordingly, we get the following obvious facts (proofs omitted), where

$$\sqrt{\sigma} = \{ i : (i, i) \in \sigma \} .$$

Remark 18. For any state $\sigma \in \Sigma_P$, where P is some finite set of propositional atoms, for any world $i \in \sqrt{\sigma}$, $\sigma[\chi_i] = \{(i, i)\}$. \(\dashv\)

Remark 19. For any $\sigma \in \Sigma_P$, P a finite set of propositional atoms, for any $i \in \sqrt{\sigma}$,

$$\sigma[\neg\chi_i] = \sigma - \{(j, k) \in \sigma : j = i \vee k = i\} . \quad \dashv$$

Remark 18 tells us that an update with a characteristic formula yields the state that contains but the reflexive pair the formula is indexed to, and Remark 19 says that updating a state with the negation of a characteristic formula eliminates all pairs from that state that involve the world the formula characterizes. We are now ready to prove the functional completeness theorem.

Theorem 20 (Functional completeness). *For all $\sigma, \sigma' \in \Sigma_P$, for P a finite set of propositional letters, such that $\sigma' \subseteq \sigma$, there is a sequence of updates τ where each formula uses only atoms, \vee and \neg , s.t. $\sigma[\tau] = \sigma'$.*

Proof. Let $\sigma' \subseteq \sigma$ be given, and put $\delta := \sigma - \sigma'$. We disregard the trivial case $\delta = \emptyset$. For each $(i, j) \in \delta$, define $\varphi_{i,j} := \neg\chi_i \vee \neg\chi_j$, and let τ be any sequence of all such $\varphi_{i,j}$.

Consider an arbitrary $\varphi_{i,j}$. By the definition of disjunction,

$$\sigma[\varphi_{i,j}] = \sigma[\neg\chi_i] \cup \sigma[\neg\chi_j]$$

and so, given Remark 19, we get that

$$\sigma[\varphi_{i,j}] = \sigma - (\{(a, b) \in \sigma : a = i \vee b = i\} \cap \{(c, d) \in \sigma : c = j \vee d = j\}) .$$

The only pairs in that intersection are of course (i, j) and (j, i) , so

$$\sigma[\varphi_{i,j}] = \sigma - \{(i, j), (j, i)\} .$$

Notice that, for the case (i, i) , we have $\sigma[\varphi_{i,i}] = \sigma[\neg\chi_i] \cup \sigma[\neg\chi_i] = \sigma[\neg\chi_i]$. So, we eliminate all pairs in σ containing i , including (i, i) .

Clearly, all and only elements from δ are eliminated by the update with the sequence τ , and thus $\sigma[\tau] = \sigma - \delta = \sigma'$. \(\square\)

The reason why I have constrained languages and models to have only finitely many propositional atoms is now clear: under my definition of functional completeness, I need formulas to be able to uniquely characterize models; equivalently, functional completeness relies on characteristic formulas' ability to uniquely identify worlds in the root of a model. If worlds become valuations over infinitely many atoms, then two worlds i and j may well be indistinguishable from each other via a given formula φ or (finite)² sequence τ . I will conclude this section on functional completeness with a proof of this proposition.

Claim 21 (Functional incompleteness). *An inquisitive language L with infinitely many propositional atoms, with a corresponding class Σ of (uncountably many) infinite models, is not functionally complete.*

Proof. We reason by contradiction. Assume L is functionally complete for Σ , the class of all possible models for a language with infinitely many propositional atoms, and consider a model σ that is the total relation on an underlying set of worlds $P \subseteq W$, where, for $n \in \mathbb{N}$ and $i_n \in W$, $i_n \in P$ iff i_n satisfies

$$i_n(p_m) = \begin{cases} 1 & \text{if } m \leq n \\ 0 & \text{otherwise} \end{cases} ;$$

that is, P is as illustrated in Table 2.1, and clearly $P \subset W$.

	i_0	i_1	i_2	i_3	\dots
p_0	1	1	1	1	\dots
p_1	0	1	1	1	\dots
p_2	0	0	1	1	\dots
p_3	0	0	0	1	\dots
\vdots	\vdots	\vdots	\vdots	\vdots	\ddots

Table 2.1: An inaccessible model

Since L is functionally complete by assumption and $\sigma \subset \omega$, for ω the maximal model in Σ , there is a (finite) sequence τ such that $\omega[\tau] = \sigma$; thus, we can conjoin all formulas in τ to get a formula φ such that $\omega[\varphi] = \sigma$. Now, let n be the largest natural number such that p_n is used in φ , and take any world $i \in W$ that survives the update with φ (i.e., $i \in \sqrt{\sigma}$). Consider now the world $j \in W$ such that $j(p_k) = i(p_k)$ for $k \leq n$, and $j(p_{n+1}) = 0$ but $j(p_\ell) = 1$ for $\ell > n + 1$. Clearly, $j \notin P$ and therefore $(j, j) \notin \sigma$. But, since j agrees with i on all propositional atoms in φ , then $(j, j) \models \varphi$, and since $(j, j) \in \omega$ and must have survived the update with φ , $(j, j) \in \sigma$. We reach contradiction, and thus there can be no τ such that $\omega[\tau] = \sigma$, so L is not functionally complete for Σ . \square

2.1.3 Alternative disjunctive normal forms

As I mentioned before, InqL is not classical logic, and it is thus no surprise that it lacks the normal form theorems we find in the proof theory of classical propositional logic. However, we can define a weaker notion of a normal form which will prove to be interesting and potentially useful.

²Naturally, this limitation disappears if we allow formulas or, perhaps more plausibly, sequences of infinite length, but I will not pursue that line here.

Definition 22 (Alternative disjunctive normal form). φ is an alternative disjunctive normal form (ADNF) iff $\varphi = \bigvee_{i < n} \varphi_i$, where each φ_i is formed exclusively with $\{\wedge, \neg, \rightarrow\}$, i.e., each φ_i is an assertion. \dashv

Alternative disjunctive normal forms are possibly large disjunctions of disjunction-free formulas so, intuitively, they bring the request for a choice to be made — remember this was our loose heuristic for disjunction in an inquisitive semantics — to the uppermost level in any given formula.

I will now prove the ADNF theorem and then proceed to a short discussion about the interest of these normal forms.³

Theorem 23 (Existence of ADNFs). *For every φ in the language, there is an ADNF $\varphi^{\vee!}$ such that $\varphi \Leftrightarrow \varphi^{\vee!}$.*

Proof. I present an algorithm that generates such $\varphi^{\vee!}$ based on φ :

1. φ is atomic. Then $\varphi^{\vee!} = \varphi$.
2. $\varphi = \psi \vee \theta$. By induction hypothesis, there are $\psi^{\vee!}$ and $\theta^{\vee!}$, so set $\varphi^{\vee!} = \psi^{\vee!} \vee \theta^{\vee!}$.
3. $\varphi = \psi \wedge \theta$. By IH, we have $\psi^{\vee!}$ and $\theta^{\vee!}$. By distributivity of disjunction over conjunction, we get that

$$\varphi \Leftrightarrow \psi \wedge \theta \Leftrightarrow \psi^{\vee!} \wedge \theta^{\vee!} \Leftrightarrow \bigvee_{i,j} (\psi_i \wedge \theta_j) .$$

Set $\varphi^{\vee!}$ to the latter formula in the equivalence above.

4. $\varphi = \neg\psi$. We assume we have $\psi^{\vee!} = \bigvee_{i < n} \psi_i$. The DeMorgan law pushing negation into disjunction / pulling negations from conjunction (namely $\neg(\alpha \vee \beta) \Leftrightarrow \neg\alpha \wedge \neg\beta$) holds, so we can set

$$\varphi^{\vee!} = \bigwedge_{i < n} \neg\psi_i ,$$

to get a trivial ADNF.

5. $\varphi = \psi \rightarrow \theta$. By induction hypothesis, we have $\psi^{\vee!}$ and $\theta^{\vee!}$. Obviously, $(\alpha \vee \beta \rightarrow \gamma) \Leftrightarrow (\alpha \rightarrow \gamma) \wedge (\beta \rightarrow \gamma)$ is valid in **lnqL** (informal argument: notice that it is an **IPL** tautology), so we have the equivalence below.

$$\psi^{\vee!} \rightarrow \theta^{\vee!} \Leftrightarrow \bigwedge_{i < n} (\psi_i \rightarrow \bigvee_{j < m} \theta_j)$$

In **lnqL**, $(\neg\alpha \rightarrow \beta \vee \gamma) \rightarrow (\neg\alpha \rightarrow \beta) \vee (\neg\alpha \rightarrow \gamma)$, the **KP** axiom, is valid, as I remarked above, as is (trivially) the other direction. I have also mentioned that the assertive fragment is negative, in the sense that assertive formulas are equivalent to their double negation; thus, given that all ψ_i are assertions, we can apply **KP** to each implication above, and so

$$\bigwedge_{i < n} (\psi_i \rightarrow \bigvee_{j < m} \theta_j) \Leftrightarrow \bigwedge_{i < n} \left(\bigvee_{j < m} (\psi_i \rightarrow \theta_j) \right) .$$

³Notice that some steps in this proof use theorems that have not been proven yet; for the most part, these are obvious validities with trivial proofs (distributivity, associativity, some directions of DeMorgan), but one or two others are more substantial results that I will prove in later sections.

Each implication is now purely assertive (for each ψ_i and θ_j is assertive by assumption), so we can formulate them in terms of conjunction and negation, via $(\alpha \rightarrow \beta) \leftrightarrow \neg(\alpha \wedge \neg\beta)$. As a consequence,

$$\bigwedge_{i < n} \left(\bigvee_{j < m} (\psi_i \rightarrow \theta_j) \right) \Leftrightarrow \bigwedge_{i < n} \left(\bigvee_{j < m} \neg(\psi_i \wedge \neg\theta_j) \right),$$

which is clearly a conjunction of ADNFs, so we can apply the law of distributivity, as in 3. above, to each pair of conjuncts.

It is clear from the algorithm that the formula $\varphi^{\vee!}$ thus constructed is an ADNF (cf. Definition 22) and that it is equivalent to the original φ . \square

At first glance, alternative disjunctive normal forms might seem to offer a proof-theoretic counterpart to the semantic notion of alternatives, as defined in Chapter 1 and repeated below, in that each disjunct in an ADNF seems to specify a semantic alternative. Conditional questions are a simple and compelling example: the formula $p \rightarrow ?q$ has the alternative set $\{p \rightarrow q, p \rightarrow \neg q\}$, and it corresponds to the ADNF $\neg(p \wedge \neg q) \vee \neg(p \wedge q)$, equivalently $(p \rightarrow q) \vee (p \rightarrow \neg q)$. If this observation is generalizable, then ADNFs give us a way to syntactically calculate alternatives, i.e., we can syntactically manipulate formulas of **lnqL** and read off those transformations the alternatives that a formula has the potential of generating in any given state.

Definition 24 (Alternatives). For σ a model of **lnqL**, the set A_σ is the set of alternatives in σ , where $\alpha \in A_\sigma$ iff

1. $\alpha \subseteq \sigma$;
2. α is a total relation; and
3. there is no $\sigma \supseteq \beta \supset \alpha$ such that β is a total relation. \dashv

Unfortunately, the correspondence between semantic alternatives and ADNFs is not as tidy as one might like. Specifically, while semantic alternatives have a maximality requirement, we cannot be sure that the ADNFs produced by the algorithm above will respect that maximality requirement. For example, there may be ADNFs of the form $\varphi \vee (\varphi \wedge \psi)$, which generate only one semantic alternative, namely φ . Naturally, we can fine-tune the ADNF algorithm to incorporate the maximality constraint, which corresponds to getting rid of disjuncts that are entailed by other disjuncts, but this involves more sophisticated manipulations than the ones above, and it becomes less clear whether ADNFs are a simpler notion than that of semantic alternatives.⁴

2.2 More on the logic **lnqL**

The remainder of this chapter will concentrate on properties of the logic (i.e., set of validities) induced by **lnqL**. I will begin by introducing a semantics more appropriate for the task at hand and will present a sound and complete axiomatization of **lnqL**.

⁴Notice that calculating semantic alternatives as in Definition 24 amounts to finding all (maximal) cliques in an undirected graph.

2.2.1 A more convenient semantics

I have already remarked that the update semantics defined here is perfectly static, as there are no real dynamic operators.⁵ Therefore, we can formulate the same inquisitive semantics in terms of static, possible world semantics, as under Definition 25 below. It is important to remark that from now onward I will drop mention to sets P of propositional atoms and will consider only one inquisitive language L , with infinitely many propositional atoms, defined in the usual way. The reason for the finiteness constraint on atoms was that we were dealing with an update system that we wanted functionally complete in the sense of Definition 16 and, in a static formulation of the semantics, functional completeness resumes its traditional definition, where finiteness of models makes no difference. The (unique) language L thus defined is interpreted in inquisitive models just as before, naturally dropping the subscripts on finite sets of propositional atoms. I will now use Σ to refer to $W \times W$, that is the set of all pairs of possible worlds. For convenience, I give the semantics of negation as a separate clause in the definition that follows, but it amounts, as before, to $\neg\varphi \iff \varphi \rightarrow \perp$, as the reader can quickly verify.

Definition 25 (Static inquisitive semantics). A formula of a language of inquisitive semantics is true at a given point $(i, j) \in \Sigma$ according to the following inductive definition:

$$\begin{aligned}
(i, j) \models p &\text{ iff } i(p) = j(p) = 1 \\
(i, j) \models \neg\varphi &\text{ iff } (i, i) \not\models \varphi \ \& \ (j, j) \not\models \varphi \\
(i, j) \models \varphi \vee \psi &\text{ iff } (i, j) \models \varphi \text{ or } (i, j) \models \psi \\
(i, j) \models \varphi \wedge \psi &\text{ iff } (i, j) \models \varphi \ \& \ (i, j) \models \psi \\
(i, j) \models \varphi \rightarrow \psi &\text{ iff } (i, i) \models \varphi \implies (i, i) \models \psi \ \& \\
&\quad (j, j) \models \varphi \implies (j, j) \models \psi \ \& \\
&\quad (i, j) \models \varphi \implies (i, j) \models \psi \quad \dashv
\end{aligned}$$

I will not show formally how Definition 25 is indeed equivalent to the update formulation, but it should be clear from merely inspecting the two definitions. The other formal notions we used are quite straightforwardly adapted to this formulation of the semantics, namely,

Definition 26 (InqL, static definition). The logic InqL is the set of formulas such that

$$\varphi \in \text{InqL} \text{ iff } (\forall (i, j) \in \Sigma) (i, j) \models \varphi \quad \dashv$$

2.2.2 An intermediate logic?

We saw before that the (proper) inclusions below hold.

$$\text{CPL} \subset \text{InqL} \subset \text{IPL}$$

⁵The issue of what makes a logic dynamic is hugely debated, so I should clarify. What I mean is that the relation of logical consequence, or entailment, as defined in the previous chapter for InqL, is perfectly standard when compared to entailment in typical dynamic semantics that have, for example, definitions of conjunction that make $\varphi \wedge \psi$ a context change potential not equivalent to $\psi \wedge \varphi$. It is in this sense that InqL is not dynamic.

Now, this suggests that **InqL** is what the literature calls an *intermediate* logic, intermediate between classical logic and intuitionistic logic, and indeed the question I'll pursue for the remainder of this section is *what* intermediate logic that might be. However, I must discuss an important caveat to what follows.

I have been calling **InqL** a *logic*, but I have not used that term in a rigorous enough way. In actuality, logics in the strict sense are typically defined as sets of sentences under a number of closure conditions, namely deduction and uniform substitution, and, as it turns out, **InqL** as we are defining it does not fall under this definition, for it fails to be closed under substitution.⁶ This is very easy to see, for

$$\neg\neg p \rightarrow p \in \mathbf{InqL} \text{ but } \neg\neg(p \vee \neg p) \rightarrow p \vee \neg p \notin \mathbf{InqL} ,$$

that is every atom is by definition assertive and thus equivalent to its double negation, but this does not carry over to arbitrary formulas; in particular, the disjunction-free fragment is classical in the sense that double negation elimination works without provisos (see Claim 28 below), but the same is not true for the full language including disjunction.

Should we be too concerned about the fact that **InqL** is not closed under substitution? Certainly not. As linguists and philosophers, we are interested in logics that can model aspects of the world in a principled manner, and if the systems that prove to be the most adequate to model natural language phenomena happen to be unorthodox from a conservative logical point of view, then that's too bad for the logician. . . This of course should not stop us from thinking about what the reason is for failure of substitution in **InqL** and what its consequences are, it is merely an idea to keep in the backs of our minds as we investigate some of the logical oddities of this instantiation of inquisitive semantics.

Clearly, **InqL** is not closed under substitution because of the validity of atomic double negation elimination, which in turn percolates to disjunction-free formulas. For the sake of clarity, I will prove this validity and consider some of its consequences.

Claim 27 (Validity of ADN). $\mathbf{InqL} \models \neg\neg p \rightarrow p$ for atoms p

Proof. Let p be given and take an arbitrary (i, j) . What we want to show is that $(i, j) \models \neg\neg p \rightarrow p$, which is equivalent to

$$\begin{aligned} (i, j) \models \neg\neg p &\implies (i, j) \models p \ \& \\ (i, i) \models \neg\neg p &\implies (i, i) \models p \ \& \\ (j, j) \models \neg\neg p &\implies (j, j) \models p . \end{aligned}$$

Now the last two conjuncts are of course true (remember reflexive points are classical), so we need only show the first conjunct. By the truth definitions above, and using again the fact that reflexive points are classical, we get

$$\begin{aligned} (i, j) \models \neg\neg p &\iff (i, i) \not\models \neg p \ \& \ (j, j) \not\models \neg p \\ &\iff (i, i) \models p \ \& \ (j, j) \models p \\ &\iff (i, j) \models p . \end{aligned} \quad \square$$

⁶But see Corollary 38 at the end of this chapter for a characterization of the logic of schematic validities of **InqL**.

And, of course, a simple inductive argument shows that disjunction-free formulas also display this property:

Claim 28 (Assertive fragment). *The disjunction-free fragment of InqL , i.e. the assertive fragment, is classical.*

Proof. It suffices to show by induction on the complexity of disjunction-free formulas that double negation elimination is valid. The base case corresponds to Claim 27, and the conjunction and implication steps use the (**IPC**) validity of pushing double negation inside conjunction and implication in connection with the induction hypothesis.⁷ \square

Interestingly enough, these two very simple results embody much of what makes InqL linguistically relevant, namely, the assumption that propositional atoms are non-inquisitive and thus purely informative, and the related intuition that it is disjunction that is entirely responsible for the existence of inquisitiveness in natural language. In my view, these two ideas are very crucial parts of an inquisitive enterprise, and thus I find the loss of closure under substitution to be by no means a good reason to consider dropping them.⁸

2.2.3 Axiomatizing InqL

Our first step towards an axiomatization of InqL is to identify a very closely related known intermediate logic, the logic **LV** of the simple forked frames (Maksimova, 1979; Chagrov and Zakharyashev, 1991).⁹

Definition 29 (The logic LV). **LV** is the logic **IPC** + (H2) + (W2), as follows:

$$\begin{aligned} \text{(H2)} \quad & \varphi \vee (\varphi \rightarrow \psi \vee \neg\psi) \\ \text{(W2)} \quad & (\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi) \vee ((\varphi \rightarrow \neg\psi) \wedge (\psi \rightarrow \neg\varphi)) . \quad \dashv \end{aligned}$$

The axioms in Definition 29 impose two conditions on frames: (H2) says that the maximum length of chains in this frame is two (where an endpoint is a chain of length one), and (W2) says that a node can have no more than two distinct successors. This means that frames of **LV** have the following shape,

⁷That is, $\neg\neg(\varphi \wedge \psi) \leftrightarrow \neg\neg\varphi \wedge \neg\neg\psi$, and similarly for implication.

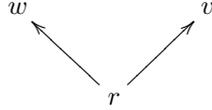
⁸The linguistic and philosophical consequence of dropping ADN would be to allow atoms to be inquisitive. Jeroen Groenendijk pointed out to me that this might be an interesting way to model *why*-questions (and see Groenendijk, 2008c, for a discussion of other consequences of dropping ADN); that may well be, but such a system would require strong meaning postulates to be usable, as it seems unlikely that having p be able to mean “It is raining. Who is the president of Austria?” can be at all useful to model *why*-questions.

⁹I thank Lex Hendriks (p.c.) for pointing out to me that **LV** had been discussed in these references. Hendriks suggested a simpler axiomatization than the one Chagrov and Zakharyashev (1991) use, namely

$$(\varphi \rightarrow \psi) \vee (\varphi \rightarrow \neg\psi) \vee (\neg\neg\psi \rightarrow \varphi) .$$

So as to keep uniformity with Chagrov and Zakharyashev’s article, I opted to keep here the axioms they use. I should also remark that Maksimova gives this logic the name L4, as part of the list of eight logics with the interpolation property.

which I will call a simple fork.



Accordingly, we have the following completeness result.

Theorem 30 (Chagrov and Zakharyashev 1991). *LV is sound and complete for the simple forked frames.*¹⁰

It is important to explicate the sense in which **LV** is close to **lnqL**. It is a fact about **lnqL** that the models needed to falsify a formula are minimal in a very precise way; this was noticed as early as Groenendijk (2007) for the logic defined there, an immediate predecessor of **lnqL**.¹¹ All we need to falsify a formula in **lnqL** is a model consisting of four pairs over two possible worlds, which Groenendijk (2007) calls a *point*:



It is easy to see why this is the case. Consider the semantic clauses as per Definition 25. The atomic clause, conjunction and disjunction clearly show that we can check whether an atom, conjunction and disjunction are true at a pair ι by simply inspecting properties of ι . Implication and negation have a broader search space, but it is still rather restricted: to check whether an implication is modeled by a pair ι , we need to see whether it is modeled at all pairs constructed from the worlds mentioned in ι , but that means, for the most complex case where $\iota = (i, j)$ with $i \neq j$, a set of four pairs, no more. As a consequence, all we need to falsify a formula in **lnqL** is at most a point as pictured above.

It is obvious what the simple forks of **LV** have in common with inquisitive points: both models have two classical nodes, namely the nodes w and v in the above picture of a fork and the pairs (i, i) and (j, j) in the depiction of a point; in addition, both models have a non-classical node, r in the **LV** fork, (i, j) (equivalently, (j, i)) in the **lnqL** point. This intuitive similarity will be explored in the proof of the Correspondence Theorem 36.

LV and **lnqL** are similar, but not identical, for atomic double negation is *not* valid in **LV**. A natural candidate for a logic that might be in exact correspondence with **lnqL** then is **LV**⁺ as defined below. The hope is that **LV**⁺ models

¹⁰In actuality, Chagrov and Zakharyashev (1991) do not prove this result, they merely cite it. The first few steps of my proof of Theorem 35 however will give an obvious outline of a proof of this theorem.

¹¹In fact, the logic of Groenendijk (2007) actually *was* **lnqL**. Groenendijk had defined an erotetic logic to address some of the issues with the partition framework discussed in Chapter 1. That logic was supposed to have purely assertive disjunction, as in a partition logic, but to allow for non-partitioning meanings such as conditional questions and disjunctions of questions. In fact though, inquisitiveness lurked in the definition of disjunction, just like in **lnqL** as defined in this thesis, except that fact had been unnoticed until I proved a functional completeness result in the spirit of the one earlier in this chapter and came across what was, in the context of that logic, a mistake. Groenendijk then had the inspired idea of proposing we pursue that non-standard definition, and inquisitive semantics was born.

will preserve the interesting shape of \mathbf{LV} models *and* validate atomic double negation. Luckily, this is indeed the case, and it is the content of Theorem 35.

Definition 31 (The logic \mathbf{LV}^+). \mathbf{LV}^+ is the “logic” $\mathbf{IPC} + (\text{H2}) + (\text{W2}) + (\text{ADN})$, as defined below:

$$\begin{aligned} (\text{H2}) \quad & \varphi \vee (\varphi \rightarrow \psi \vee \neg\psi) \\ (\text{W2}) \quad & (\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi) \vee ((\varphi \rightarrow \neg\psi) \wedge (\psi \rightarrow \neg\varphi)) \\ (\text{ADN}) \quad & \neg\neg p \rightarrow p, \text{ for all atoms } p. \end{aligned} \quad \dashv$$

Notice that \mathbf{LV}^+ is \mathbf{IPC} with all substitution instances of (H2) and (W2), but (ADN) *only for atoms*; this property of the axiom (ADN) is what explains that \mathbf{InqL} isn’t closed under uniform substitution, as discussed earlier, and it justifies the scare quotes around the word ‘logic’ in Definition 31.

(ADN) imposes a condition on *models*, as opposed to frames. In words, (ADN) says that whenever all the endpoints accessible from a node w force an atom p , then w too forces p . Equivalently, the set of atoms forced by w is the intersection of the sets of atoms forced by the endpoints accessible from w . I will dub this the *intersection property*, as follows.

Definition 32 (Intersection property). Let v be the function that maps a world in a Kripke model to the set of propositional atoms forced at that world. That is, for $\mathfrak{K} = (W, R, V)$ a Kripke model of intuitionistic logic, $v : W \rightarrow \wp(P)$ such that

$$v(x) = \{p : x \in V(p)\} .$$

A Kripke model $\mathfrak{K} = (W, R, V)$ for intuitionistic logic has the *intersection property* just in case, for any $w \in W$,

$$v(w) = \bigcap \{v(x) : wRx \ \& \ x \text{ is an endpoint}\} . \quad \dashv$$

The following simple claim will be useful in proving soundness later.

Claim 33. *Atomic double negation is valid in all finite models with the intersection property.*

Proof. Let a finite model $\mathfrak{M} = (W, R, V)$ with the intersection property be given, and assume that an arbitrary node w forces $\neg\neg p$, for some atom p , to show that $w \Vdash p$. Since \mathfrak{M} is finite, there are endpoints accessible from w , and by definition all endpoints for w model p , which means that

$$p \in \bigcap \{v(x) : wRx \ \& \ x \text{ is an endpoint}\} ,$$

but then by the intersection property $p \in v(w)$, and thus $w \Vdash p$. □

I now define the canonical model of \mathbf{LV}^+ and proceed to showing the main result in this chapter.

Definition 34 (Canonical model for \mathbf{LV}^+). The canonical model for \mathbf{LV}^+ is the tuple $\mathfrak{M} = \langle W, \subseteq, V \rangle$, wherein W is the set of all consistent \mathbf{LV}^+ theories with the disjunction property, and V is such that, for any atom p , $V(p) = \{\Gamma \in W : p \in \Gamma\}$. □

Theorem 35. \mathbf{LV}^+ is sound and complete for the simple forks with the intersection property.

Proof. (H2) and (W2) are valid in the simple forked models, and since the intersection property does not tamper with these conditions on height and width, they are also valid in the simple forks with the intersection property. From Claim 33 we know that (ADN) is also valid in such models. This gives us soundness.

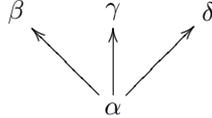
I proceed to completeness. The Lindenbaum-style lemma needed is proven in the perfectly standard way, as well as the fact that, in the canonical model, $\Gamma \Vdash \varphi$ just in case $\varphi \in \Gamma$. This gives us the model existence lemma, whereby, if $\mathbf{LV}^+ \not\vdash \varphi$, there is a \mathbf{LV}^+ -theory Γ with the disjunction property such that, in the canonical model of Definition 34, $\Gamma \not\vdash \varphi$. We must now show that the canonical model is a simple forked model with the intersection property, or that one can be obtained from it.

First, \mathfrak{M} has height at most 2. Suppose there is a chain $\alpha \subset \beta \subset \gamma \in W$. Because $\beta \subset \gamma$, there must be some $\psi \in \gamma$ such that $\psi \notin \beta$, and similarly some $\theta \in \beta$ such that $\theta \notin \alpha$. Since α forces all instances of the axiom (H2), we have

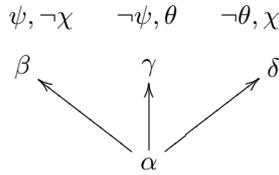
$$\alpha \Vdash \theta \vee (\theta \rightarrow \psi \vee \neg\psi),$$

but $\alpha \not\vdash \theta$, so it must be that $\alpha \Vdash \theta \rightarrow \psi \vee \neg\psi$. Now, since $\alpha \subset \beta$ and $\beta \Vdash \theta$, we have that $\beta \Vdash \psi \vee \neg\psi$. But we know that $\beta \not\vdash \psi$, and it also cannot be that $\beta \Vdash \neg\psi$, given that $\gamma \Vdash \psi$. We reach contradiction, and thus it must be that all chains in \mathfrak{M} have height at most two.

Second, any point in \mathfrak{M} has at most two (strict) successors. Suppose otherwise, i.e., there are $\alpha \subset \beta, \gamma, \delta \in W$. We know that no chain can have height greater than 2, so the model must look like the following picture.



Now, β, γ and δ must be distinct, so there must be some ψ such that $\beta \Vdash \psi$ but $\gamma \not\vdash \psi$. Notice that β, γ and δ are endpoints (given the height boundedness of these models just shown), so we also have that $\gamma \Vdash \neg\psi$. Similarly, we find $\theta \in \gamma, \theta \notin \delta$ and $\chi \in \beta, \chi \notin \delta$. This gives us the following picture.



Now, α forces all substitution instances of (W2), so we have the following.

- (1) $\alpha \Vdash (\psi \wedge \neg\chi) \vee (\chi \wedge \neg\theta) \rightarrow (\neg\psi \wedge \theta) \vee (\chi \wedge \neg\theta)$ or
- (2) $\alpha \Vdash (\neg\psi \wedge \theta) \vee (\chi \wedge \neg\theta) \rightarrow (\psi \wedge \neg\chi) \vee (\chi \wedge \neg\theta)$ or
- (3) $\alpha \Vdash (\psi \wedge \neg\chi) \vee (\chi \wedge \neg\theta) \rightarrow \neg((\neg\psi \wedge \theta) \vee (\chi \wedge \neg\theta)) \wedge$
 $(\neg\psi \wedge \theta) \vee (\chi \wedge \neg\theta) \rightarrow \neg((\psi \wedge \neg\chi) \vee (\chi \wedge \neg\theta))$

I will show that the above cannot be true.¹²

- (1) $\beta \Vdash \psi \wedge \neg\chi$, so $\beta \Vdash (\psi \vee \neg\chi) \vee (\chi \wedge \neg\theta)$. But $\beta \Vdash \psi$ and $\beta \Vdash \neg\chi$, so $\beta \not\Vdash (\neg\psi \wedge \theta) \vee (\chi \wedge \neg\theta)$, which falsifies (1).
- (2) Falsified at γ , for reasons similar to the above.
- (3) $\delta \Vdash (\psi \wedge \neg\chi) \vee (\chi \wedge \neg\psi)$, so by (3) we get the following (recall that δ is an endpoint, and therefore classical equivalences hold).

$$\begin{aligned}
& \delta \Vdash \neg((\neg\psi \wedge \theta) \vee (\chi \wedge \neg\theta)) \\
\iff & \delta \Vdash \neg(\neg\psi \wedge \theta) \wedge \neg(\chi \wedge \neg\theta) \\
\iff & \delta \Vdash (\psi \vee \neg\theta) \wedge (\neg\chi \vee \theta) \\
\implies & \delta \Vdash \neg\chi \vee \theta
\end{aligned}$$

But $\delta \not\Vdash \neg\chi \vee \theta$, so we reach contradiction.

(1), (2) and (3) are therefore false at α , which contradicts our hypothesis that α could have three distinct strict successors.¹³

It remains to show that \mathfrak{M} has the intersection property. For α the root of \mathfrak{M} , what we have to show is that

$$p \in \alpha \text{ iff } p \in \bigcap \{\beta : \alpha \subset \beta\},$$

for atoms p in the language. The crucial direction is of course right-to-left (for left-to-right follows immediately from persistence in Kripke models), so assume as an absurd hypothesis that there is an atom p such that $p \in \bigcap \{\beta : \alpha \subset \beta\}$ but $p \notin \alpha$. Obviously, $\beta \Vdash p$, for all $\beta \supset \alpha$, which entails that $\alpha \Vdash \neg p$. Since \mathfrak{M} is a model of \mathbf{LV}^+ , $\alpha \Vdash \neg p \rightarrow p$. But this means $\alpha \Vdash p$ and therefore $p \in \alpha$, which contradicts our earlier assumption.

I have shown that every node in \mathfrak{M} has depth at most 2, has at most 2 successors, and has the intersection property. Strictly speaking, \mathfrak{M} isn't just a single simple fork, but rather a collection of unconnected simple forks (cf. Figure 2.1). We get the desired simple fork by taking the generated submodel under the theory Γ that falsifies φ . That submodel is a simple fork with the intersection property. \square

¹²It might be worth spelling out the intuition behind this step. The properties of the canonical model allowed us to find formulas ψ , θ and χ that we can use in conjunction to single out each of β , γ and δ . Now, the disjunction in (1), (2), (3) is a substitution instance of the axiom (W2), repeated below, where τ is replaced by $(\psi \wedge \neg\chi) \vee (\chi \wedge \neg\theta)$ and σ by $(\neg\psi \wedge \theta) \vee (\chi \wedge \neg\theta)$.

$$(W2) \quad (\tau \rightarrow \sigma) \vee (\sigma \rightarrow \tau) \vee ((\tau \rightarrow \neg\sigma) \wedge (\sigma \rightarrow \neg\tau))$$

Since $\psi \wedge \neg\chi$ is only true at β , $\neg\psi \wedge \theta$ at γ and $\chi \wedge \neg\theta$ at δ , what the formula in (1) reads is “if you are at β or δ then you are at γ or δ ,” which of course is falsifiable at β . Similarly for (2).

(3) then reads “if you are at β or δ then you are not at γ or δ , and if you are at γ or δ then you are not at β or δ ,” which is false at δ .

¹³This much suffices for completeness, but what in fact holds is something stronger: any node is either a single reflexive point or has *exactly* two successors. To show this, it suffices to prove that a node in the canonical model cannot have exactly one successor. Suppose that were the case, that is, we have $\alpha \subset \beta$ and no $\gamma \neq \beta$ such that $\alpha \subset \gamma$. There must be some atom $\varphi \in \beta$ such that $\varphi \notin \alpha$, and since β is the only successor of α , this must be some atom p ; but by the intersection property it must hold that $p \in \alpha$, a contradiction.

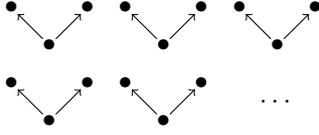
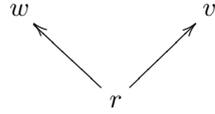


Figure 2.1: The canonical model for \mathbf{LV}^+

Theorem 36 (Correspondence). $\mathbf{InqL} \models \varphi$ iff $\mathbf{LV}^+ \models \varphi$

Proof. We will show the converse, namely, that $\mathbf{InqL} \not\models \varphi$ iff $\mathbf{LV}^+ \not\models \varphi$, which amounts to building countermodels of one logic from countermodels of the other.

Let us begin with left-to-right, so assume we have $\mathbf{InqL} \not\models \varphi$. By definition, this means we have some σ , a model of \mathbf{InqL} , with $(i, j) \in \sigma$ such that $\sigma, (i, j) \not\models \varphi$. Now construct a Kripke model for intuitionistic logic $\mathfrak{M} = (W, R, V)$, with $W = \{r, w, v\}$, and R and V as defined below.



$$\begin{aligned} w \in V(p) &\text{ iff } i(p) = 1 \\ v \in V(p) &\text{ iff } j(p) = 1 \\ r \in V(p) &\text{ iff } i(p) = j(p) = 1 \end{aligned}$$

\mathfrak{M} is a simple fork and by the definition of V above it clearly has the intersection property, so it is a model of \mathbf{LV}^+ . We now show by induction on the complexity of formulas that $w \Vdash \psi$ iff $(i, i) \models \psi$:

1. the base case and \perp follow immediately from the definition of V .
2. conjunction (and similarly disjunction):

$$\begin{aligned} w \Vdash \psi \wedge \theta &\iff w \Vdash \psi \ \& \ w \Vdash \theta \\ \text{(by the IH)} &\iff (i, i) \models \psi \ \& \ (i, i) \models \theta \\ &\iff (i, i) \models \psi \wedge \theta \end{aligned}$$

3. implication (remember that w and (i, i) are both endpoints in their respective models):

$$\begin{aligned} w \Vdash \psi \rightarrow \theta &\iff w \Vdash \psi \implies w \Vdash \theta \\ \text{(by IH)} &\iff (i, i) \models \psi \implies (i, i) \models \theta \\ &\iff (i, i) \models \psi \rightarrow \theta \end{aligned}$$

Obviously, we also have that $v \Vdash \psi$ iff $(j, j) \models \psi$, so call these two results an intermediate lemma for this proof. We can now show, again by induction, that $r \Vdash \psi$ iff $(i, j) \models \psi$. Base case, conjunction and disjunction are quite trivial, the

interesting step is implication:

$$\begin{aligned}
r \Vdash \psi \rightarrow \theta &\iff (\forall x : rRx) x \Vdash \psi \implies x \Vdash \theta \\
\text{(given the nature of } \mathfrak{M}) &\iff r \Vdash \psi \implies r \Vdash \theta \ \& \\
&w \Vdash \psi \implies w \Vdash \theta \ \& \\
&v \Vdash \psi \implies v \Vdash \theta \\
\text{(by IH and the lemma above)} &\iff (i, j) \Vdash \psi \implies (i, j) \Vdash \theta \ \& \\
&(i, i) \Vdash \psi \implies (i, i) \Vdash \theta \ \& \\
&(j, j) \Vdash \psi \implies (j, j) \Vdash \theta \\
&\iff (i, j) \Vdash \psi \rightarrow \theta
\end{aligned}$$

As a consequence of this, $\mathfrak{M}, r \not\models \varphi$, and as such $\mathbf{LV}^+ \not\models \varphi$, as we intended to show.

For the other direction, assume $\mathbf{LV}^+ \not\models \varphi$, which gives us an \mathbf{LV}^+ model $\mathfrak{M} = (W, R, V)$, such that, for r its root, $\mathfrak{M}, r \not\models \varphi$. Now, \mathfrak{M} , as a model of \mathbf{LV}^+ , can be taken to be a simple fork with the intersection property; the fact that it is a simple fork tells us it must fall under one of three possible configurations:

1. \mathfrak{M} is a single reflexive point. Then φ can be falsified in classical logic, and it is trivial to build an \mathbf{InqL} model that falsifies it as well: simply take a single reflexive point (i, i) that models exactly the same atoms as the r in \mathfrak{M} .
2. \mathfrak{M} is of the form $r \rightarrow w$. Since \mathfrak{M} has the intersection property, clearly r and w must force the same propositional atoms, and so this case reduces to the single reflexive point. Proceed as in 1. above.
3. \mathfrak{M} is indeed a fork with root r and endpoints w and v .

For the latter case, define $\sigma := \{(i, i), (j, j), (i, j), (j, i)\}$, and let, for each p in the language,

$$\begin{aligned}
i(p) &= 1 \text{ iff } w \in V(p) \\
j(p) &= 1 \text{ iff } v \in V(p) .
\end{aligned}$$

It should be clear from the way we defined i that $(i, i) \Vdash \psi$ iff $w \Vdash \psi$, and similarly for j and v . We proceed then to show by induction that $(i, j) \Vdash \psi$ iff $r \Vdash \psi$:

1. \perp is trivial, so consider just the atomic case:

$$\begin{aligned}
(i, j) \Vdash p &\iff i(p) = j(p) = 1 \\
&\iff w \in V(p) \ \& \ v \in V(p) \\
\text{(by the intersection property)} &\iff r \in V(p) \\
&\iff r \Vdash p
\end{aligned}$$

2. conjunction and disjunction are trivial.

3. implication uses the fact that $(i, i) \vDash \psi$ iff $w \Vdash \psi$, and similarly for j and v , the induction hypothesis and the makeup of \mathfrak{M} , just as a few paragraphs above in the proof of the other direction of this theorem, so I'll omit it in the interest of space.

Just as before, this gives us, since we have the assumption $\mathfrak{M}, r \not\vDash \varphi$, that $\sigma, (i, j) \not\vDash \varphi$, and thus $\text{lnqL} \not\vDash \varphi$, as we wanted to show. \square

As I mentioned before, Theorem 36 gives us a sound and complete axiomatization of lnqL , as it legitimizes merely importing the axioms of \mathbf{LV}^+ . Accordingly, we get the following immediate corollary of Theorems 36 and 35:

Corollary 37 (Completeness of lnqL). $\text{IPC} + (\text{H2}) + (\text{W2}) + (\text{ADN})$ for atomic p is a complete axiomatization of lnqL .

Finally, I draw attention to the tight connection between the logics \mathbf{LV} and $\mathbf{LV}^+ / \text{lnqL}$: \mathbf{LV} is the logic of the schematic validities of \mathbf{LV}^+ and lnqL , that is, \mathbf{LV} is the set of all formulas that are valid in \mathbf{LV}^+ and lnqL for all substitution instances.

Corollary 38 (Schematic validities of \mathbf{LV}^+ and lnqL). \mathbf{LV} is the logic of the schematic validities of \mathbf{LV}^+ (and therefore of lnqL).

Proof. We must show that, for any formula $\varphi \notin \mathbf{LV}$, there is a uniform substitution instance φ' of φ such that $\varphi' \notin \mathbf{LV}^+$. Clearly, the non-trivial cases of φ are those that capitalize on the validity of atomic double negation, so it suffices to show the above for all atoms. Take a simple fork \mathfrak{M} of \mathbf{LV} that falsifies $\varphi := \neg\neg p \rightarrow p$ for an arbitrary atom p , and consider the model \mathfrak{M}' of \mathbf{LV}^+ that is exactly like \mathfrak{M} except in that it has the intersection property (in particular, the formula $\neg\neg p \rightarrow p$ is valid). There must be a formula ψ that is forced by one endpoint but not the other, respectively a formula θ . Both endpoints therefore satisfy $\psi \vee \theta$, but the root does not. Now substitute $\psi \vee \theta$ for p in φ , and let this be φ' . The root of \mathfrak{M}' clearly falsifies $\varphi' = \neg\neg(\psi \vee \theta) \rightarrow \psi \vee \theta$. \square

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