Bounded rationality and logic for epistemic modals[1]

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Abstract. Kratzer (1991) provides *comparative epistemic modals* such as '*at least as likely as*' with their models in terms of a *qualitative ordering*. Yalcin (2010) shows that Kratzer's model does *not validate* some intuitively *valid* inference schemata and *validates* some intuitively *invalid* ones. He adopts a model based directly on a *probability measure* for comparative epistemic modals. His model does *not* cause this problem. However, as Kratzer (2012) says, Yalcin's model seems to be *unnatural* as a model for comparative epistemic modals. Holliday and Icard (2013) prove that not only a *probability measure model* but also a *qualitatively additive measure model* and a *revised version of Kratzer's model* do not cause Yalcin's problem. Suzuki (2013) proposes a logic the model of which reflects Kratzer's intuition above, does not cause Yalcin's problem, and has *no limitation of the size* of the domain. In the models of Holliday and Icard (2013) and Suzuki (2013), the *transitivity* of probabilistic indifference is valid. The transitivity of probabilistic indifference can lead to a *sorites paradox*. The *nontransitivity* of probabilistic indifference can be regarded as a manifestation of *bounded rationality*. The *aim* of this paper is to propose a new version of *complete* logic—Ḃoundedly-Rational L̇ogic for Ėpistemic Modals (BLE)—the model of the language of which has the following *three* merits: (1) The model reflects *Kratzer's intuition* above in the sense that the model should not be based directly on probability measures, but based on qualitative probability orderings. (2) The model does not cause *Yalcin's problem*. (3) The model is *boundedly-rational* in the sense that the transitivity of probabilistic indifference is not valid. So it does not invite the *sorites paradox*.

Keywords: bounded rationality, epistemic modal, just noticeable difference, modal logic, representation theorem, semiordered qualitative probability, sorites paradox

1. Motivation

Kratzer (1991) provides *comparative epistemic modals* such as '*at least as likely as*' with their models in terms of a *qualitative ordering* on propositions derived from a qualitative ordering on possible worlds. Yalcin (2010) shows that Kratzer's model does *not validate* some intuitively *valid* inference schemata and *validates* some intuitively *invalid* ones. He adopts a model based directly on a *probability measure* for comparative epistemic modals. His model does *not* cause this problem. However, as Kratzer (2012) says, 'Our semantic knowledge alone does not give us the *precise quantitative notions of probability and desirability* that mathematicians and scientists work with', Yalcin's model seems to be *unnatural* as a model for comparative epistemic modals. Holliday and Icard (2013) prove that not only a *probability measure model* but also a *qualitatively additive measure model* and a *revised version of Kratzer's model* do not cause Yalcin's problem. Suzuki (2013) proposes a logic the model of which reflects Kratzer's intuition above, does not cause Yalcin's problem, and has *no limitation of the size* of the domain.

Generally, the standard models of social sciences are based on *global rationality* that requires

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an *optimising behavior*. But according to Simon (1982a, b, 1997), cognitive and information-processing constrains on the capabilities of agents, together with the complexity of their environment, render an optimising behavior an *unattainable ideal*. Simon dismisses the idea that agents should exhibit global rationality and suggests that they in fact exhibit *bounded rationality* that allows a *satisficing behavior*. If an agent has only a *limited* ability of discrimination, he may be considered to be only *boundedly rational*. In the models of Holliday and Icard (2013) and Suzuki (2013), the *transitivity* of probabilistic indifference is valid. The following example shows that the transitivity of probabilistic indifference can lead to a *sorites paradox*:

Example 1 (Sorites Paradox) *Suppose that a prep-school has 1000 candidates, and that a staff member of the school arranges them in order of the average of examination results:* $c_1(top), c_2, \ldots, c_{1000}(bottom)$*, and that, for any* $i(1 \leq i \leq 999)$*,* c_i *will pass the university entrance exam* as likely as c_{i+1} *for him, and that* c_1 *will pass it by far* more likely than c_{1000} *for him. Then if* probabilistic indifference *were* transitive*,* c_1 *would result in passing it* as likely as c_{1000} *for the staff member.*

The *nontransitivity* of probabilistic indifference can be regarded as a manifestation of *bounded rationality*. An agent has only a limited ability of discrimination. The psychophysicist Fechner (1860) explains this limited ability by the concept of a *threshold of discrimination*, that is, *just noticeable difference* (*JND*). Given a measure function f that an examiner could assign to a boundedly rational examinee for an object a, its JND δ is the *lowest intensity increment* such that $f(a) + \delta$ is recognized to be higher than $f(a)$ by the examinee. We can consider a JND from a *probabilistic* point of view. Domotor and Stelzer (1971) introduce the concept of *semiordered qualitative probability* that can provide a qualitatively probabilistic counterpart of a JND.

The *aim* of this paper is to propose a new version of *complete* logic—Ḃoundedly-Rational L̇ogic for Ėpistemic Modals (BLE)—the model of the language of which has the following *three* merits:

1. The model reflects *Kratzer's intuition* above in the sense that the model should not be based directly on probability measures, but based on qualitative probability orderings.

2. The model does not cause *Yalcin's problem*.

3. The model is *boundedly-rational* in the sense that transitivity of probabilistic indifference is not valid. So it does not invite the *sorites paradox* in Example 1.

The structure of this paper is as follows. In Section 2, we show a representation theorem by Domotor and Stelzer (1971) related to a normalized JND. In Subsection 3.1, we define the language $\mathscr{L}_{\mathsf{BLE}}$ of BLE. In Subsection 3.2, we define a structured model $\mathfrak{M}$ of $\mathscr{L}_{\mathsf{BLE}}$, provide BLE with a truth definition at $w \in W$ in $\mathfrak{M}$, define the truth in $\mathfrak{M}$, define validity, provide BLE with some truth conditions in terms of a probability measure, justify the (in)validity of Yalcin's formulae in BLE, and show the invalidity of the transitivity of probabilistic indifference in BLE. In Subsection 3.3, we provide BLE with its proof system. In Subsection 3.4, we show the soundness and completeness theorems of BLE. In Section 4, we finish with brief concluding remarks.

2. Representation theorem for $\succ$

Domotor and Stelzer (1971) prove the following theorem in which δ is interpreted to mean a *normalized JND*:

Theorem 1 (Representation Theorem for $\succ$, Domotor and Stelzer (1971)) *Suppose that W is a nonempty* finite *set of possible worlds, and that $\mathscr{F}$ is the Boolean algebra of subsets of W, and that $\succ$ is a binary relation on $\mathscr{F}$. Then there exists a finitely additive probability measure $P : \mathscr{F} \to \mathbb{R}$ and $\delta \in \mathbb{R}$ satisfying*

$$A \succ B \text{ iff } P(A) \geq P(B) + \delta,$$

where $0 < \delta \leq 1$ iff the following conditions are met:

1. ***Nontriviality**: $W \succ \emptyset$.*
2. ***Irreflexivity**: Not $(A \succ A)$, for any $A \in \mathscr{F}$.*
3. ***Dominance**: For any $A, B, C \in \mathscr{F}$, if $A \subseteq B$, then if $C \succ B$, then $C \succ A$.*
4. ***Semi-Scottness**: For any $n \geq 1$ and any $A_1, \ldots, A_n, B_1, \ldots, B_n, C_1, \ldots, C_n, D_1, \ldots, D_n \in \mathscr{F}$, if for any $i < n$, $(A_i \succ B_i$ and not $(C_i \succ D_i))$, then if $A_n \succ B_n$, then $C_n \succ D_n$, given that*

$$\bigcup_{1 \leq i_1 < \cdots < i_k \leq n} ((A_{i_1} \cup D_{i_1}) \cap \cdots \cap (A_{i_k} \cup D_{i_k})) = \bigcup_{1 \leq i_1 < \cdots < i_k \leq n} ((B_{i_1} \cup C_{i_1}) \cap \cdots \cap (B_{i_k} \cup C_{i_k}))$$

   holds for any k with $1 \leq k \leq n$.

Remark 1 (Semi-Scottness) *Intuitively, the part after 'given that' of* Semi-Scottness *means that for any $w \in W$, w is in exactly as many $A_i \cup D_i$'s as $B_i \cup C_i$'s.*

3. Boundedly-rational Logic for Epistemic Modals (BLE)

3.1. Language

We define the language $\mathscr{L}_{\mathsf{BLE}}$ of BLE as follows:

Definition 1 (Language) *Let $\mathscr{S}$ denote a set of sentential variables, $\Box$ a unary sentential operator, and $>$ a binary sentential operator. The language $\mathscr{L}_{\mathsf{BLE}}$ of* BLE *is given by the following BNF grammar:*

$$\varphi ::= s \mid \top \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi > \varphi) \mid \Box\varphi$$

such that $s \in \mathscr{S}$.

- $\bot, \vee, \rightarrow$ *and* $\leftrightarrow$ *are introduced by the standard definitions.*
- $\varphi > \psi$ *is interpreted to mean that* φ *is* more likely than ψ.
- $\varphi \geqslant \psi := \neg(\psi > \varphi)$.
- $\varphi \geqslant \psi$ *is interpreted to mean that* φ *is* at least as likely as ψ.
- $\varphi \approx \psi := \neg(\varphi > \psi) \wedge \neg(\psi > \varphi)$.
- $\varphi \approx \psi$ *is interpreted to mean that* φ *is* as likely as ψ.
- $\triangle\varphi := \varphi > \neg\varphi$.
- $\triangle\varphi$ *is interpreted to mean that* probably φ.
- $\Box\varphi$ *is interpreted to mean that* it must be that φ.
- $\Diamond\varphi := \neg\Box\neg\varphi$.
- $\Diamond\varphi$ *is interpreted to mean that* it might be that φ.

3.2. Semantics

We define a *structured model* $\mathfrak{M}$ of $\mathscr{L}_{\mathsf{BLE}}$ as follows:

Definition 2 (Model) $\mathfrak{M}$ *is a quadruple* (W, R, ρ, V) *in which*

- W *is a non-empty set of possible worlds,*
- R *is a binary epistemic accessibility relation on* W*,*
- ρ *is a* finitely additive semiordered qualitative probability space assignment *that assigns to each* $w \in W$ *a finitely additive semiordered qualitative probability space* $(W_w, \mathscr{F}_w, \succ_w)$ *in which*
  - $W_w := \{w' \in W : R(w, w')\}$,
  - $\mathscr{F}_w$ *is the Boolean algebra of subsets of* W_w *with* $\emptyset$ *as zero element and* W_w *as unit element, and*
  - $\succ_w$ *is a finitely additive semiordered qualitative probability ordering relative to* $w \in W$ *on* $\mathscr{F}_w$ *that satisfies all of* Nontriviality, Irreflexivity, Dominance, *and* Semi-Scottness *of Theorem 1, and*

- *V is a truth assignment to each $s \in \mathcal{S}$ for each $w \in W$.*

We provide BLE with the following truth definition at $w \in W$ in $\mathfrak{M}$, define the truth in $\mathfrak{M}$, and then define validity as follows:

Definition 3 (Truth and Validity) *The notion of $\varphi \in \Phi_{\mathscr{L}_{\mathsf{BLE}}}$ being true at $w \in W$ in $\mathfrak{M}$, in symbols $(\mathfrak{M}, w) \models_{\mathscr{L}_{\mathsf{BLE}}} \varphi$, is inductively defined as follows:*

- $(\mathfrak{M}, w) \models_{\mathscr{L}_{\mathsf{BLE}}} s$ *iff* $V(w)(s) = \text{true}$.
- $(\mathfrak{M}, w) \models_{\mathscr{L}_{\mathsf{BLE}}} \top$.
- $(\mathfrak{M}, w) \models_{\mathscr{L}_{\mathsf{BLE}}} \neg\varphi$ *iff* $(\mathfrak{M}, w) \not\models_{\mathscr{L}_{\mathsf{BLE}}} \varphi$.
- $(\mathfrak{M}, w) \models_{\mathscr{L}_{\mathsf{BLE}}} \varphi \wedge \psi$ *iff* $(\mathfrak{M}, w) \models_{\mathscr{L}_{\mathsf{BLE}}} \varphi$ *and* $(\mathfrak{M}, w) \models_{\mathscr{L}_{\mathsf{BLE}}} \psi$.
- $(\mathfrak{M}, w) \models_{\mathscr{L}_{\mathsf{BLE}}} \varphi > \psi$ *iff* $[\![\varphi]\!]^{\mathfrak{M}}_w \succ_w [\![\psi]\!]^{\mathfrak{M}}_w$, *where* $[\![\varphi]\!]^{\mathfrak{M}}_w := \{w' \in W_w : (\mathfrak{M}, w') \models_{\mathscr{L}_{\mathsf{BLE}}} \varphi\}$.
- $(\mathfrak{M}, w) \models_{\mathscr{L}_{\mathsf{BLE}}} \Box\varphi$ *iff for any w' such that $R(w, w')$,* $(\mathfrak{M}, w') \models_{\mathscr{L}_{\mathsf{BLE}}} \varphi$.

If $(\mathfrak{M}, w) \models_{\mathscr{L}_{\mathsf{BLE}}} \varphi$ for any $w \in W$, we write $\mathfrak{M} \models_{\mathscr{L}_{\mathsf{BLE}}} \varphi$ and say that φ is true in $\mathfrak{M}$. If φ is true in all models of $\mathscr{L}_{\mathsf{BLE}}$, we write $\models_{\mathscr{L}_{\mathsf{BLE}}} \varphi$ and say that φ is valid.

The next corollary follows from Definitions 1 and 3.

Corollary 1 (Truth Condition of $\varphi \approx \psi$ and Truth Condition of $\triangle\varphi$)

- $(\mathfrak{M}, w) \models_{\mathscr{L}_{\mathsf{BLE}}} \varphi \approx \psi$ *iff* $[\![\varphi]\!]^{\mathfrak{M}}_w \sim_w [\![\psi]\!]^{\mathfrak{M}}_w$, *where* $[\![\varphi]\!]^{\mathfrak{M}}_w \sim_w [\![\psi]\!]^{\mathfrak{M}}_w :=$ *not* $([\![\varphi]\!]^{\mathfrak{M}}_w \succ_w [\![\psi]\!]^{\mathfrak{M}}_w)$ *and not* $([\![\psi]\!]^{\mathfrak{M}}_w \succ_w [\![\varphi]\!]^{\mathfrak{M}}_w)$.
- $(\mathfrak{M}, w) \models_{\mathscr{L}_{\mathsf{BLE}}} \triangle\varphi$ *iff* $[\![\varphi]\!]^{\mathfrak{M}}_w \succ_w \overline{[\![\varphi]\!]^{\mathfrak{M}}_w}$.

Then the next corollary follows from Theorem 1 and Corollary 1.

Corollary 2 (Truth Conditions by Probability Measure) *For any $w \in W$, there exists $P_w : \mathscr{F} \to \mathbb{R}$ and such δ that $0 < \delta \leq 1$ satisfying*

- $(\mathfrak{M}, w) \models_{\mathscr{L}_{\mathsf{BLE}}} \varphi > \psi$ *iff* $P_w([\![\varphi]\!]^{\mathfrak{M}}_w) \geq P_w([\![\psi]\!]^{\mathfrak{M}}_w) + \delta$.
- $(\mathfrak{M}, w) \models_{\mathscr{L}_{\mathsf{BLE}}} \varphi \approx \psi$ *iff* $P_w([\![\psi]\!]^{\mathfrak{M}}_w) - \delta < P_w([\![\varphi]\!]^{\mathfrak{M}}_w) < P_w([\![\psi]\!]^{\mathfrak{M}}_w) + \delta$.
- $(\mathfrak{M}, w) \models_{\mathscr{L}_{\mathsf{BLE}}} \triangle\varphi$ *iff* $P_w([\![\varphi]\!]^{\mathfrak{M}}_w) \geq \dfrac{1+\delta}{2}$.

Remark 2 (Logic of Inexact Knowledge) *In* BLE *the truth clause of the epistemic necessity* *operator* $\Box$ *is not based on a semiordered qualitative probability ordering. In* BLE *the truth clause of* $\Box\varphi$ *is given in Definition 3 as follows:*

$$(\mathfrak{M},w) \models_{\mathscr{L}_{\mathsf{BLE}}} \Box\varphi \quad \textit{iff}, \quad \textit{for any } w' \textit{ such that } R(w,w'), \quad (\mathfrak{M},w') \models_{\mathscr{L}_{\mathsf{BLE}}} \varphi.$$

*On the other hand, Suzuki (2016) proposes a new version of complete logic—L̇ogic of İnexact K̇nowledge (*LIK*)—the model of the language of which can reflect Williamson (1994)'s arguments on inexact knowledge in the sense that the truth condition of the knowledge operator* K ($K\varphi := \varphi \approx \top$) *is given in terms of a semiordered qualitative probability ordering as follows:*

$$(\mathfrak{M},w) \models_{\mathscr{L}_{\mathsf{LIK}}} K\varphi \quad \textit{iff} \quad [\![\varphi]\!]^{\mathfrak{M}} \sim_w W.$$

So, by virtue of Theorem 1, for any $w \in W$*, there exists* $P_w : \mathscr{F} \to \mathbb{R}$ *and such* δ *that* $0 < \delta \leq 1$ *satisfying*

$$(\mathfrak{M},w) \models_{\mathscr{L}_{\mathsf{LIK}}} K\varphi \quad \textit{iff} \quad 1-\delta < P_w([\![\varphi]\!]^{\mathfrak{M}}) \leq 1.$$

We can also construct BLE *on the basis of this idea.*

Yalcin (2010) presents the following list of *intuitively valid* formulae (V1)–(V11) and *intuitively invalid* formulae (I1) and (I2):

- (V1) $\triangle\varphi \to \neg\triangle\neg\varphi$.

  (If probably φ, then it is not probable that not φ.).

- (V2) $\triangle(\varphi \wedge \psi) \to (\triangle\varphi \wedge \triangle\psi)$.

  (If probably (φ and ψ), then (probably φ and probably ψ).)

- (V3) $\triangle\varphi \to \triangle(\varphi \vee \psi)$.

  (If probably φ, then probably (φ or ψ).)

- (V4) $\varphi \succcurlyeq \bot$.

  (φ is at least as likely as $\bot$.)

- (V5) $\top \succcurlyeq \varphi$.

  ($\top$ is at least as likely as φ.)

- (V6) $\Box\varphi \to \triangle\varphi$.

  (If it must be that φ, then probably φ.)

- (V7) $\triangle\varphi \to \Diamond\varphi$.

  (If probably φ, then it might be that φ.)

- (V8) $(\varphi \to \psi) \to (\triangle\varphi \to \triangle\psi)$.

  (If (if φ, then ψ), then (if probably φ, then probably ψ).)

- (V9) $(\varphi \to \psi) \to (\neg\triangle\psi \to \neg\triangle\varphi)$.

  (If (if φ, then ψ), then (if it is not probable that ψ, then it is not probable that φ).)

- (V10) $(\varphi \to \psi) \to (\psi \geqslant \varphi)$.

  (If (if φ, then ψ), then (ψ is at least as likely as φ).)

- (V11) $(\psi \geqslant \varphi) \to (\triangle\varphi \to \triangle\psi)$.

  (If (ψ is at least as likely as φ), then (if probably φ, then probably ψ).)

- (V12) $(\psi \geqslant \varphi) \to ((\varphi \geqslant \neg\varphi) \to (\psi \geqslant \neg\psi))$.

  (If (ψ is at least as likely as φ), then (if (φ is at least as likely as not φ), then (ψ is at least as likely as not ψ)).)

- (I1) $((\varphi \geqslant \psi) \wedge (\varphi \geqslant \chi)) \to (\varphi \geqslant (\psi \vee \chi))$.

  (If ((φ is at least as likely as ψ) and (φ is at least as likely as χ)), then (φ is at least as likely as (ψ or χ)).)

- (I2) $(\varphi \approx \neg\varphi) \to (\varphi \geqslant \psi)$.

  (If (φ is as likely as not φ), then (φ is at least as likely as ψ).)

We justify the (in)validity of Yalcin's formulae in BLE as follows:

Proposition 1 (Justification of Yalcin's Formulae) BLE *validates all of (V1)–(V12) and validate neither (I1) nor (I2).*

Moreover, in BLE, the transitivity of probabilistic indifference is not valid:

Proposition 2 (Invalidity of Transitivity of Probabilistic Indifference)

$\not\models_{\mathscr{L}_{\mathsf{BLE}}} ((\varphi \approx \psi) \wedge (\psi \approx \chi)) \to (\varphi \approx \chi)$.

So the *sorites paradox* in Example 1 does not appear in BLE.

3.3. Syntax

The proof system of BLE consists of the following:

Definition 4 (Proof System)

Axioms

- *All tautologies of classical sentential logic,*
- $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ *(K),*
- $(\Box(\varphi_1 \leftrightarrow \varphi_2) \wedge \Box(\psi_1 \leftrightarrow \psi_2)) \to ((\varphi_1 > \psi_1) \leftrightarrow (\varphi_2 > \psi_2))$ *(Replacement of Known Equivalents on >),*
- $\top > \bot$ *(Syntactic Counterpart of **Nontriviality**),*
- $\neg(\varphi > \varphi)$ *(Syntactic Counterpart of **Irreflexivity**), and*
- $$\left(\bigvee_{1 \le i_1 < \cdots < i_k \le n} ((\varphi_{i_1} \vee \tau_{i_1}) \wedge \cdots \wedge (\varphi_{i_k} \vee \tau_{i_k})) \leftrightarrow \bigvee_{1 \le i_1 < \cdots < i_k \le n} ((\psi_{i_1} \vee \chi_{i_1}) \wedge \cdots \wedge (\psi_{i_k} \vee \chi_{i_k})) \right) \to \left(\bigwedge_{i=1}^{n-1} ((\varphi_i > \psi_i) \wedge \neg(\chi_i > \tau_i)) \to ((\varphi_n > \psi_n) \to (\chi_n > \tau_n)) \right),$$
  for any $n \ge 1$ and any k with $1 \le k \le n$

  *(Syntactic Counterpart of **Semi-Scottness**).*

Inference Rules

- $\dfrac{\varphi \to \psi}{(\chi > \psi) \to (\chi > \varphi)}$ *(Syntactic Counterpart of **Dominance**),*
- *Modus Ponens, and*
- *Necessitation.*

A proof of $\varphi \in \Phi_{\mathscr{L}_{\mathsf{BLE}}}$ is a finite sequence of $\mathscr{L}_{\mathsf{BLE}}$-formulae having φ as the last formula such that either each formula is an instance of an axiom or it can be obtained from formulae that appear earlier in the sequence by applying an inference rule. If there is a proof of φ, we write

$\vdash_{\mathsf{BLE}} \varphi$.

Remark 3 (Infinite Schema) *The syntactic counterpart of Semi-Scottness is an* infinite schema *of axioms for any $n \geq 1$ and any k with $1 \leq k \leq n$.*

3.4. Metalogic

On the basis of Segerberg (1971) and Gärdenfors (1975), we can prove the soundness and completeness theorems of BLE:

Theorem 2 (Soundness) *For any $\varphi \in \Phi_{\mathscr{L}_{\mathsf{BLE}}}$, if $\vdash_{\mathsf{BLE}} \varphi$, then $\models_{\mathscr{L}_{\mathsf{BLE}}} \varphi$.*

Theorem 3 (Completeness) *For any $\varphi \in \Phi_{\mathscr{L}_{\mathsf{BLE}}}$, if $\models_{\mathscr{L}_{\mathsf{BLE}}} \varphi$, then $\vdash_{\mathsf{BLE}} \varphi$.*

4. Concluding Remarks

In this paper, we have proposed a new version of *complete* logic—Ḃoundedly-Rational L̇ogic for Ėpistemic Modals (BLE)—the model of the language of which has the following *three* merits:

1. The model reflects *Kratzer's intuition* above in the sense that the model is not based directly on probability measures, but based on qualitative probability orderings.

2. The model does not cause *Yalcin's problem*.

3. The model is *boundedly-rational* in the sense that the transitivity of probabilistic indifference is not valid. So it does not invite the *sorites paradox* in Example 1.

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