Interactive Semantic Alignment Model: Social Influence and Local Transmission Bottleneck (technical appendix)

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1 Introduction

In this technical appendix we provide proof techniques and proofs of the representation theorems from [Kalociński et al., 2018]. In general, the idea behind our proofs is easy. Hopefully, this manuscript will help the reader to quickly grasp how these and similar results can be achieved. Note that this manuscript is not self-contained. Its main purpose is to help the reader to understand the technical content of [Kalociński et al., 2018].

2 General idea

We are given a set of agents A (in our case, $A = \{1, 2\}$), a set of hypotheses H (in our case, $H = F_k$ for some $k > 0$, where F_k is the set of all completely reduced fractions whose denominator do not exceed k) and a random variable X with an associated probability function P, assuming values from the $[0, 1]$ interval. We are also given a function reflecting the social impact of each agent, $w : A \to \mathbb{R}_+$ (in our case, these are just two numbers: w_1 for agent 1 and w_2 for agent 2). Finally, we are given a "bottleneck" parameter n—a natural number indicating how many stimuli appear per interaction stage (in our case, $n = 1$ or $n = 2$).

Let us recall briefly how coordination proceeds. At each step $t = 0, 1, \ldots$, every agent uses some hypothesis from H which is captured by the so-called synchronic description $s_t : A \to H$. During the interaction stage (within a given step), n stimuli are drawn from X: $r_1, \ldots, r_n$. Each agent $i \in A$ makes a public announcement consisting of n truth values $v_1, \ldots, v_n$ where $v_j =$ the truth value of "$r_j > s_t(i)$". Next, agents align (alignment stage) using the alignment operator (consult the paper). In our case, for example, the input of the alignment operator, as executed by agent 1, consists of three lists: $(w_2, \ldots, w_2)$, $(r_1, \ldots, r_n)$, $(v_1, \ldots, v_n)$ where the first list is simply the authority of agent 2 repeated n times, $r_1, \dots, r_n$ are the generated stimuli and $v_1, \dots, v_n$ are the public truth values generated by agent 2. The corresponding input lists for agent 2 would be $(w_1, \dots, w_1)$, $(r_1, \dots, r_n)$ and $(v'_1, \dots, v'_n)$, where the difference lies in the list of authorities (agent 2 takes into account the authority of his interlocutor, agent 1, hence w_1) and in the list of truth values $v'_1, \dots, v'_n$ which were generated by agent 1. After executing the alignment operator, agents change their hypotheses accordingly which leads to the synchronic description s_{t+1} and the whole process starts anew at step $t+1$ with s_{t+1} as the new basis for the generation of truth values.

As noted in the paper, coordination thus described is a Markov process. We would like to compute the transition probabilities $p_{uv \to u'v'}$ for all $u, v, u', v' \in H$. Recall that $p_{uv \to u'v'}$ is the probability that a dyad changes its synchronic description from uv (agent 1 using u, agent 2 using v) to $u'v'$.

First, let $n = 1$. This means that during an interaction stage only one stimulus is drawn from X. Let $uv \in H^2$ be a synchronic description. Denote by $M^{uv}_{a,r}$ the set of hypotheses which get the maximal value of the reward function (consult the paper) computed relative to agent $a \in \{1, 2\}$, the state $uv \in H^2$, and the stimulus $r \in [0, 1]$. To calculate transition probabilities to other states (including uv), we have to compute $M^{uv}_{a,r}$, for all $r \in [0, 1]$, $a \in A$. This might seem as a daunting task, as there are uncountably many such sets. In our case, this can be done quite easily, as each hypothesis corresponds to a certain point in $[0, 1]$ and accepts all stimuli that exceed it (hypothesis 1 is an exception here, as hypothesis 1 accepts 1, see the paper for details). So whenever you have two hypotheses that do not have any other hypotheses from H in between, you are sure that stimuli lying in between would yield the same truth value (*true* for all hypotheses on the left and *false* for all hypotheses on the right), and thus the reward function would remain fixed for this interval for all hypotheses. Since we consider finite sets of hypotheses, this way you can always get a finite partition of the sort we want. So, eventually, we can obtain a partition the sample space of stimuli $[0, 1]$ into mutually disjoint events $E_1, \dots, E_k$ satisfying two conditions:

1. $\bigcup_{i=1}^{k} E_i = [0, 1]$ and
2. for all $1 \leq i \leq k$, for each $a \in A$ and for all $r, r' \in E_i$, $M^{uv}_{a,r} = M^{uv}_{a,r'}$.[1]

One caveat that you should bear in mind is that the reward function depends on the state of the agent. Essentially, his current hypothesis always gets extra reward points equal to his authority.[2] Hence, just to be safe, if we calculate transitions from the state $uv \in H^2$, we always carve out two special sets E, E' which are included in the final list $E_1, E_2, \dots, E_k$, namely $E = \{u\}$, $E' = \{v\}$.

Suppose $E_1, \dots, E_k$ are as desired. This allows us to denote by M^{uv}_{a,E_i} the set of hypotheses which obtain a maximal reward relative to agent a and the

[1] Obviously, there will be many such partitions as you can always refine a partition satisfying these conditions without violating the requirements. We will usually stick to relatively small partitions.

[2] This is to reflect a sort of inner bias to preserve one's current state.

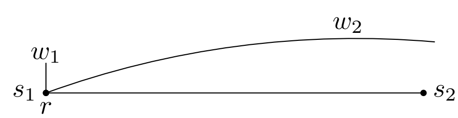
state uv. In other words, M^{uv}_{a,E_i} consists of all hypotheses which would get the agent a the highest reward (during interaction), provided that the population is in the state uv and the stimulus $\in E_i$.

Now, let $E'_1, \ldots, E'_m$ be all events among $E_1, \ldots, E_k$, such that $xy \in M^{uv}_{1,E'_j} \times M^{uv}_{2,E'_j}$, for $j = 1, 2, \ldots, m$. Then $p_{uv \to xy}$ is calculated by summing up, for $j = 1, 2, \ldots, m$: the probability of getting E'_j multiplied by the probability of changing from uv to xy provided the stimulus is drawn from E'_j (the latter probability depends on the notion of simplicity over hypotheses—consult the paper).

3 Bottleneck-only Condition

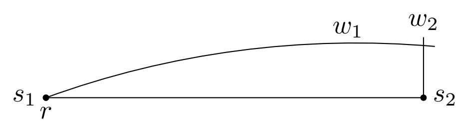
3.1 State 01

3.1.1 $r = 0$



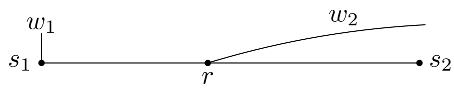
$M^{01}_{1,r=0} = \{0\}$

The graph above represents a unit interval $[0, 1]$ with 0 on the left and 1 on the right. s_1 represent the hypothesis of agent 1, s_2 of agent 2. We want to compute $M^{01}_{1,r=0}$. So, agent 1 uses 0 ($s_1 = 0$) and agent 2 uses 1 ($s_2 = 1$) as their current hypotheses. We calculate which hypotheses get the maximal reward according to agent 1 ($_1$ subscript). W consider event $E = \{0\}$, i.e. we consider the situation when the stimulus $r = 0$. Agent 1 recognizes that his interlocutor's response to r is *false* (since the truth value of $s_2 > r$ is *false* when $r = 0$). Which hypotheses from H would yield *false* for $r = 0$? It turns out that these are all hypotheses from H. This is marked in the graph by drawing an arc starting from 0 and going over the whole interval. Above the arc we see w_2 which means that every hypothesis lying under this arc is rewarded with w_2 points. However, agent 1 also looks at his own current hypothesis which is 0 and he rewards it with w_1 points. This fact is marked in the graph by drawing a vertical line from 0 and the number of points above it. As you can see, the hypothesis 0 is rewarded with an overall number of points $w_1 + w_2$ whereas other hypotheses with w_2 which yields $M^{01}_{1,r=0} = \{0\}$.

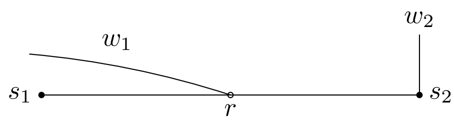


$M^{01}_{2,r=0} = \{1\}$

3.1.2 $r \in (0, 1)$

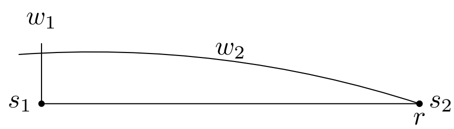


$M^{01}_{1,r\in(0,1)} = \{0\} \cup \{h \geq r\}$

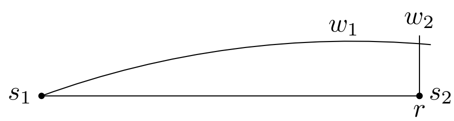


$M^{01}_{2,r\in(0,1)} = \{h < r\} \cup \{1\}$

3.1.3 $r = 1$



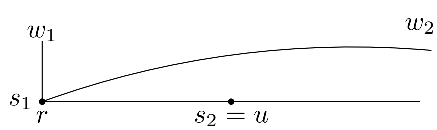
$M^{01}_{1,r=1} = \{0\}$



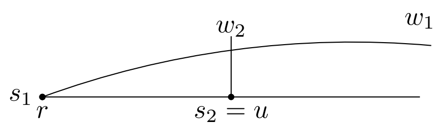
$M^{01}_{2,r=1} = \{1\}$

3.2 State $0u$, $0 < u < 1$

3.2.1 $r = 0$

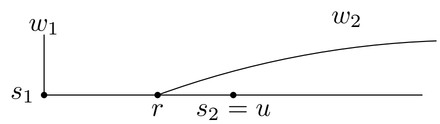


$M^{0u}_{1,r=0} = \{0\}$

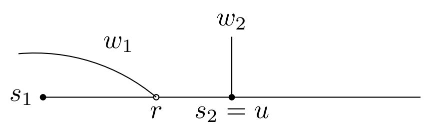


$M^{0u}_{2,r=0} = \{u\}$

3.2.2 $r \in (0, u]$

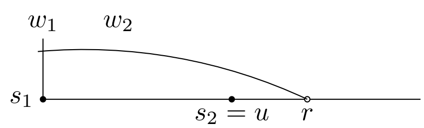


$$M^{0u}_{1,r\in(0,u]} = \{0\} \cup \{h \geq r\}$$

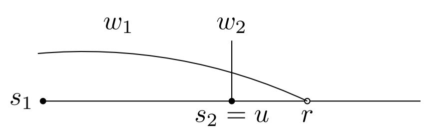


$$M^{0u}_{2,r\in(0,u]} = \{h < r\} \cup \{u\}$$

3.2.3 $r \in (u, 1)$

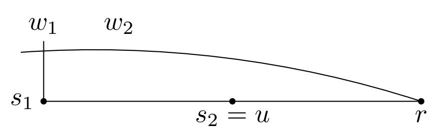


$$M^{0u}_{1,r\in(u,1)} = \{0\}$$

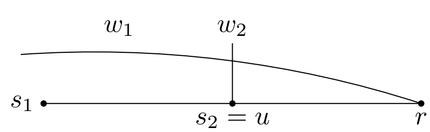


$$M^{0u}_{2,r\in(u,1)} = \{u\}$$

3.2.4 $r = 1$



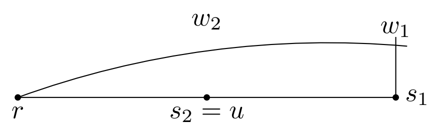
$$M^{0u}_{1,r=1} = \{0\}$$



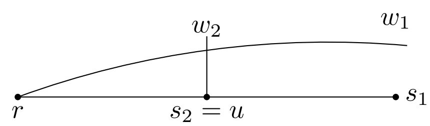
$$M^{0u}_{2,r=1} = \{u\}$$

3.3 State $1u$, $0 < u < 1$

3.3.1 $r = 0$

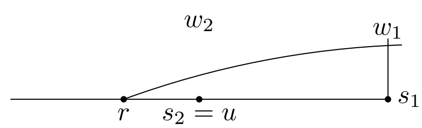


$M^{1u}_{1,r=0} = \{1\}$

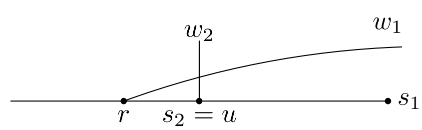


$M^{1u}_{2,r=0} = \{u\}$

3.3.2 $r \in (0, u]$

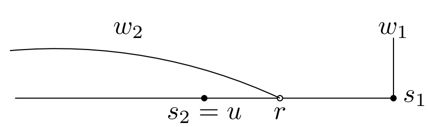


$M^{1u}_{1,r\in(0,u]} = \{1\}$

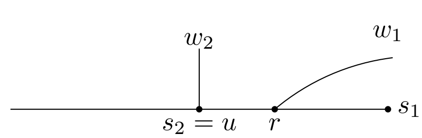


$M^{1u}_{2,r\in(0,u]} = \{u\}$

3.3.3 $r \in (u, 1)$

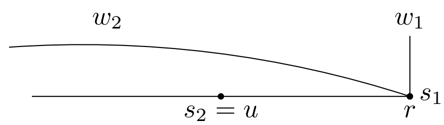


$M^{1u}_{1,r\in(u,1)} = \{h < r\} \cup \{1\}$

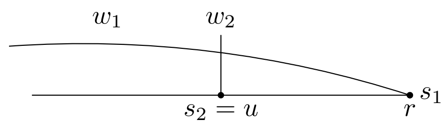


$M^{1u}_{2,r\in(u,1)} = \{u\} \cup \{h \geq r\}$

3.3.4 $r = 1$



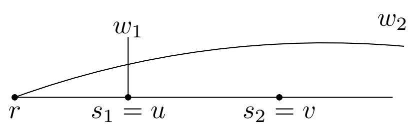
$M^{1u}_{1,r=1} = \{1\}$



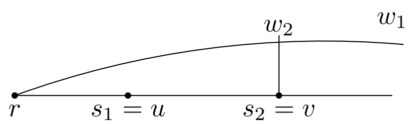
$M^{1u}_{2,r=1} = \{u\}$

3.4 State uv, $0 < u < v < 1$

3.4.1 $r = 0$, $(w_1 = w_2)$

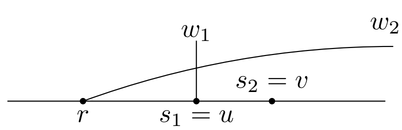


$M^{uv}_{1,r=0} = \{u\}$

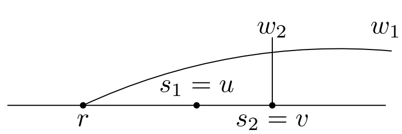


$M^{uv}_{2,r=0} = \{v\}$

3.4.2 $r \in (0, u]$, $(w_1 = w_2)$

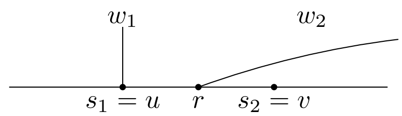


$M^{uv}_{1,r\in(0,u]} = \{u\}$

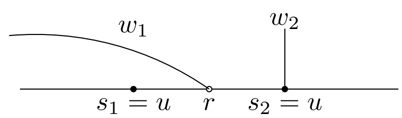


$M^{uv}_{2,r\in(0,u]} = \{v\}$

3.4.3 $r \in (u, v]$, $(w_1 = w_2)$

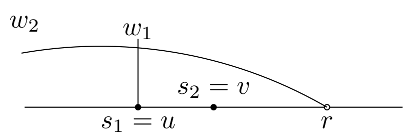


$$M^{uv}_{1,r\in(0,u]} = \{u\} \cup \{h \geq r\}$$

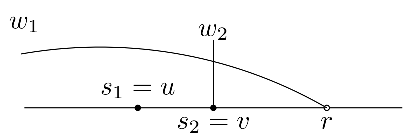


$$M^{uv}_{2,r\in(u,v]} = \{h < r\} \cup \{v\}$$

3.4.4 $r \in (v, 1)$, $(w_1 = w_2)$

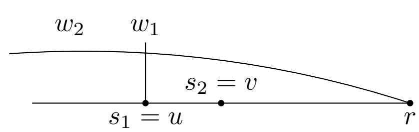


$$M^{uv}_{1,r\in(v,1)} = \{u\}$$

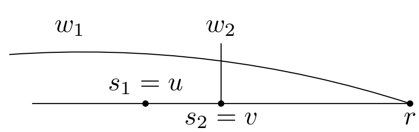


$$M^{uv}_{2,r\in(v,1)} = \{v\}$$

3.4.5 $r = 1$, $(w_1 = w_2)$



$$M^{uv}_{1,r=1} = \{u\}$$



$$M^{uv}_{2,r=1} = \{u\}$$

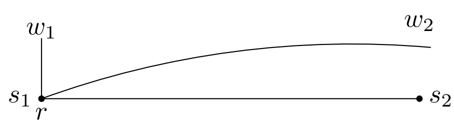
4 Authority-and-Bottleneck Condition, $w_1 > w_2$

Observe that there is no need to draw graphs for agent 1 as he will always stick to his current hypothesis. It follows directly from $w_1 > w_2$ and $n = 1$. Agent 1 assigns reward w_1 to his current meaning. If some other meanings are rewarded, their value is w_2 which is less than w_1. The current hypothesis of agent 1 could also be rewarded additional w_2 points—but then his current hypothesis has even greater reward, namely $w_1 + w_2$ which is also greater than w_2 which some other hypotheses might get.

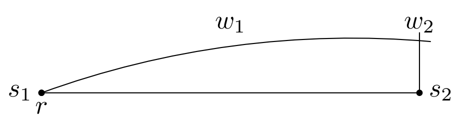
We draw the graphs for agent 1 only for the state 01 so that the reader can see how it works.

4.1 State 01

4.1.1 $r = 0$, $(w_1 > w_2)$

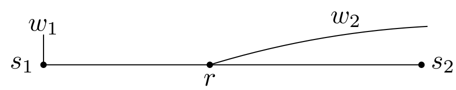


$$M^{01}_{1,r=0} = \{0\}$$

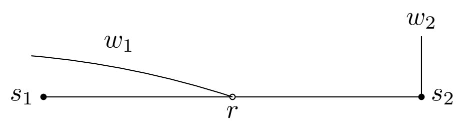


$$M^{01}_{2,r=0} = \{1\}$$

4.1.2 $r \in (0,1)$, $(w_1 > w_2)$

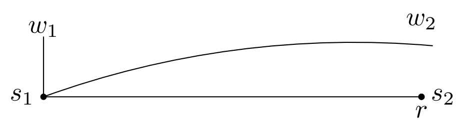


$$M^{01}_{1,r\in(0,1)} = \{0\}$$

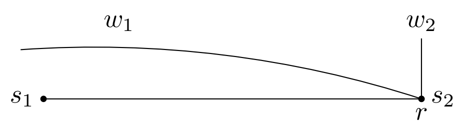


$$M^{01}_{2,r\in(0,1)} = \{h < r\}$$

4.1.3 $r = 1$, $(w_1 > w_2)$



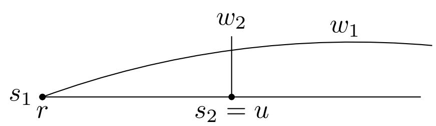
$$M^{01}_{1,r=1} = \{0\}$$



$$M^{01}_{1,r=1} = \{1\}$$

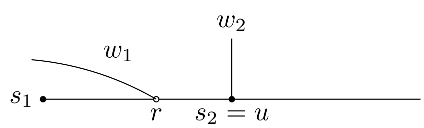
4.2 State $0u$, $0 < u < 1$.

4.2.1 $r = 0$, $(w_1 > w_2)$



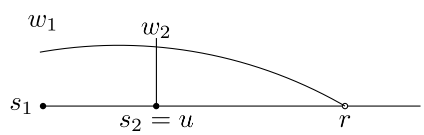
$M^{0u}_{2,r=0} = \{u\}$

4.2.2 $r \in (0, u]$, $(w_1 > w_2)$



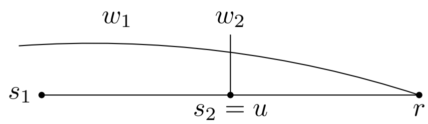
$M^{0u}_{2,r\in(0,u]} = \{h < r\}$

4.2.3 $r \in (u, 1)$, $(w_1 > w_2)$



$M^{0u}_{2,r\in(u,1)} = \{u\}$

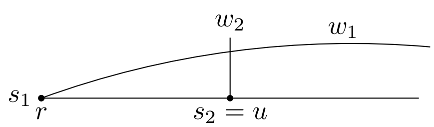
4.2.4 $r = 1$, $(w_1 > w_2)$



$$M^{0u}_{2,r=1} = \{u\}$$

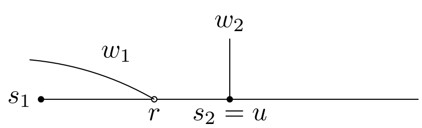
4.3 State $0u$, $0 < u < 1$.

4.3.1 $r = 0$, $(w_1 > w_2)$



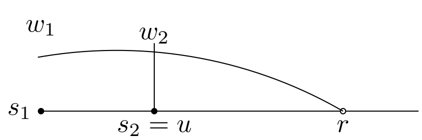
$$M^{0u}_{2,r=0} = \{u\}$$

4.3.2 $r \in (0, u]$, $(w_1 > w_2)$



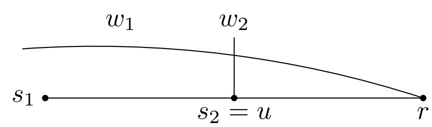
$$M^{0u}_{2,r\in(0,u]} = \{h < r\}$$

4.3.3 $r \in (u, 1)$, $(w_1 > w_2)$



$$M^{0u}_{2,r\in(u,1)} = \{u\}$$

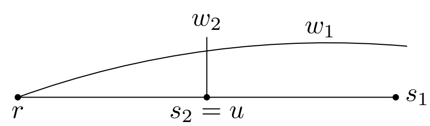
4.3.4 $r = 1$, $(w_1 > w_2)$



$$M^{0u}_{2,r=1} = \{u\}$$

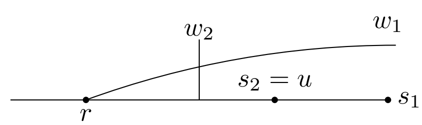
4.4 State $1u$, $0 < u < 1$.

4.4.1 State $1u$, $r = 0$, $(w_1 > w_2)$



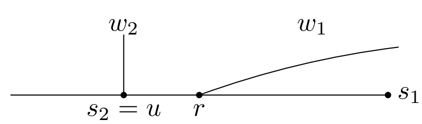
$$M^{1u}_{2,r=0} = \{u\}.$$

4.4.2 State $1u$, $r \in (0, u]$, $(w_1 > w_2)$



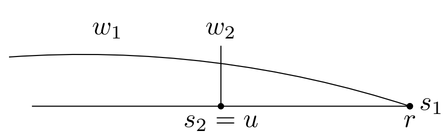
$$M^{1u}_{2,r\in(0,u]} = \{u\}$$

4.4.3 State $1u$, $r \in (u, 1)$, $(w_1 > w_2)$



$$M^{1u}_{2,r\in(u,1)} = \{h \geq r\}$$

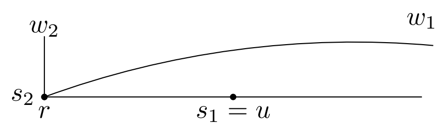
4.4.4 State $1u$, $r = 1$, $(w_1 > w_2)$



$$M^{1u}_{2,r=1} = \{u\}$$

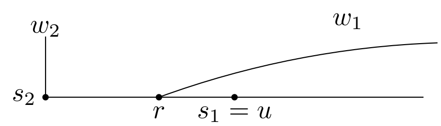
4.5 State $u0$, $0 < u < 1$.

4.5.1 State $u0$, $r = 0$, $(w_1 > w_2)$



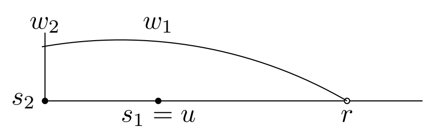
$$M^{u0}_{2,r=0} = \{0\}$$

4.5.2 State $u0$, $r \in (0, u]$, $(w_1 > w_2)$



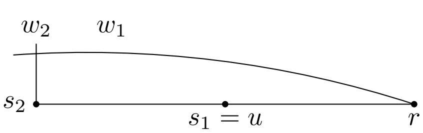
$$M^{u0}_{2,r\in(0,u]} = \{h \geq r\}$$

4.5.3 State $u0$, $r \in (u, 1)$, $(w_1 > w_2)$



$$M^{u0}_{2,r\in(u,1)} = \{0\}$$

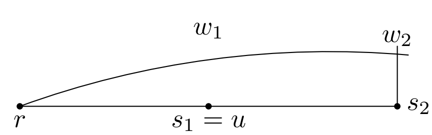
4.5.4 State $u0$, $r = 1$, $(w_1 > w_2)$



$$M^{u0}_{2,r=1} = \{0\}$$

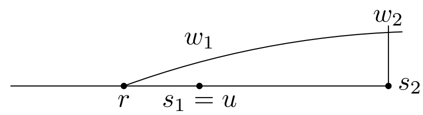
4.6 State $u1$, $0 < u < 1$.

4.6.1 State $u1$, $r = 0$, $(w_1 > w_2)$



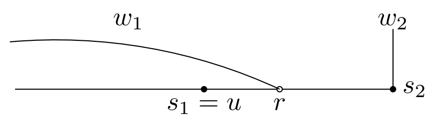
$$M^{u1}_{2,r=0} = \{1\}$$

4.6.2 State $u1$, $r \in (0, u]$, $(w_1 > w_2)$



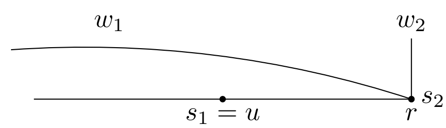
$$M^{u1}_{2,r\in(0,u]} = \{1\}$$

4.6.3 State $u1$, $r \in (u, 1)$, $(w_1 > w_2)$



$$M^{u1}_{2,r\in(u,1)} = \{h < r\}$$

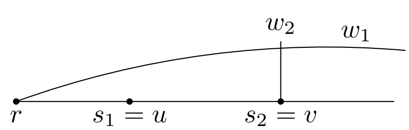
4.6.4 State $u1$, $r = 1$, $(w_1 > w_2)$



$$M^{u1}_{2,r=1} = \{1\}$$

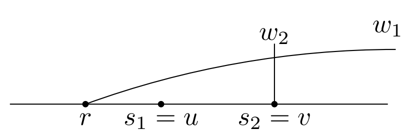
4.7 State uv, $0 < u < v < 1$.

4.7.1 State uv, $r = 0$, $(w_1 > w_2)$



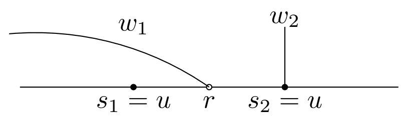
$$M^{uv}_{2,r=0} = \{v\}.$$

4.7.2 State uv, $r \in (0, u]$, $(w_1 > w_2)$



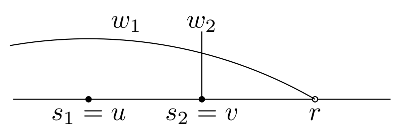
$$M^{uv}_{2,r\in(0,u]} = \{v\}$$

4.7.3 State uv, $r \in (u, v]$, $(w_1 > w_2)$



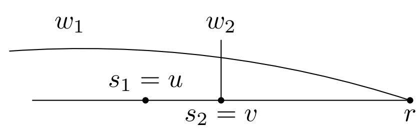
$$M^{uv}_{2,r\in(u,v]} = \{h < r\}$$

4.7.4 State uv, $r \in (v, 1)$, $(w_1 > w_2)$



$$M^{uv}_{2,r\in(v,1)} = \{v\}$$

4.7.5 State uv, $r = 1$, $(w_1 > w_2)$



$$M^{uv}_{2,r=1} = \{v\}$$

References

[Kalociński et al., 2018] Kalociński, D., Mostowski, M., and Gierasimczuk, N. (2018). Interactive semantic alignment model: social influence and local transmission bottleneck. *Journal of Logic, Language and Information.*