October 14, 2009 **Not the final draft, guaranteed to change**

Free choice permission as resource-sensitive reasoning[1]

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Abstract. Free choice permission is a long-standing puzzle in deontic logic and natural language semantics involving a so-called conjunctive use of *or*: from *You may eat an apple* ***or*** *a pear*, we can infer that *You may eat an apple* and that *You may eat a pear*—but not that *You may eat an apple* ***and*** *a pear*. Following Lokhorst 1997, I argue that because permission is a limited resource, a resource-sensitive logic such as Girard's Linear Logic is better suited to modeling permission talk than, say, classical logic. A resource-sensitive approach enables the semantics to track not only that permission has been granted and what sort of permission it is (whether to eat apples or to eat pears), but also *how much* permission has been granted, i.e., whether there is enough permission to eat two pieces of fruit or only one. In addition, I suggest that *or* is ambiguous between a familiar disjunctive use (translating to Linear Logic's additive disjunction, $\oplus$), and a use on which it forces a choice between equally viable but competing alternatives (Linear Logic's additive conjunction, &). The account here is primarily semantic, with no special modes of composition or special pragmatic rules. The paper includes an introduction to the core ideas of Linear Logic.

Since Ross 1941, it has been clear that the logic of obligation and permission behaves dramatically differently than ordinary reasoning:

(1) a. You may eat an apple or a pear.
    b. You may eat an apple.
    c. You may eat a pear.

If (1a) is true, then it is certainly true that you may eat an apple. Likewise, it is equally true that you have it within your power to safely eat a pear. So

[1]Thanks for helpful discussions with Simon Charlow, Emmanuel Chemla, Cleo Condoravdi, Judith Degen, Nicholas Fleisher, Sven Lauer, Paul Portner, Daniel Rothschild, Philippe Schlenker, Chung-chieh Shan, and Seth Yalcin.

an adequate account of the meaning of (1a) must explain how it comes to entail (1b) and (1c).

This entailment pattern is by no means the usual case. Consider a variation on (1) in which the permissive modal *may* is omitted:

(2) a. You ate an apple or a pear.
b. You ate an apple.
c. You ate a pear.

In this case, (2a) certainly does not entail either (2b) or (2c). So something about permission talk correlates with the unusual implications we are concerned with here.

The puzzle posed by the facts in (1) is known as the free choice permission problem (Kamp 1973 attributes the choice of name to von Wright).

Since (1a) entails both (1b) and (1c), (1b) and (1c) are therefore both true simultaneously. Crucially, however, (1a) does not give you permission to eat more than one piece of fruit.

(3) a. You may eat an apple and you may (also) eat a pear.
b. You may eat an apple or you may (*also) eat a pear.

Thus in many discussions, (1a) is said to entail (3a), since (3a) is merely the conjunction of (1b) and (1c). Note, however, that (3a) has an interpretation on which it furnishes permission to eat more than one piece of fruit. This interpretation is the one compatible with adding *also* in the second conjunct. Now, although (1a) is consistent with a situation in which the addressee is allowed to eat more than one piece of fruit (as we will see below), the truth of (1a) alone is never sufficient to guarantee that more than one piece of fruit may be eaten. As a result, (3b) is a better candidate for a paraphrase of (1a): it, too (surprisingly!) entails (1b) and (1c), but, like (1a), it too does not ever justify eating more than one piece of fruit. This is why *also* is never appropriate in the second disjunct in (3b) (except on an irrelevant reading on which the prejacent of *also* includes *may*).

What I am suggesting is that a complete characterization of permission sentences must not only tell us whether permission exists and what type of permission it is (i.e., permission to eat an apple versus permission to eat a pear), it must also characterize *how much* permission has been granted. Thus it must predict that (1a) and (3b) guarantee permission only to eat one piece of fruit, but that (2a) can be used to provide permission to eat two pieces of fruit.

The key insight that I would like to develop in this paper is due to Lokhorst 1997: that permission and obligation is a resource-sensitive domain, so that logics based on classical propositional logic are not appropriate. Lokhorst suggests using Girard's 1987 Linear Logic instead, and I will follow the technical details of his proposal closely. The contribution of this paper will be to introduce Lokhorst's work to a linguistic audience, to evaluate it with respect to competing linguistic analyses, and to investigate the implications of adapting Lokhorst's proposal for the theory of natural language semantics and pragmatics. In addition, I will propose that *or* is ambiguous between a disjunctive use (Linear Logic's additive disjunction, $\oplus$) and a use in which it forces a choice between competing alternatives (Linear Logic's additive conjunction, &). These uses align roughly with classical disjunction and with the so-called "exclusive *or*", but in both cases are logically distinct.

Resource-sensitive ('substructural') logics are already familiar in linguistics as tools for building syntax/semantics interfaces (e.g., Moortgat 1997 or Dalrymple 2001). As far as I know, however, no one has yet suggested that natural language connectives such as *or* or *and* can have uses in which they behave semantically like connectives in a substructural logic, as I am suggesting here.

Kamp 1973, 1978 discusses free choice permission not just as a puzzle for reasoning about obligation (deontic logic), but as a puzzle for the meaning of natural language expressions. From the point of view of natural language semantics, the interesting thing about the free choice permission problem is that it appears to require not only making assumptions about the meaning of certain uses of modal expressions such as *may*, but about the meaning of the corresponding uses of the coordinating conjunctions *and* and *or*. This will be true of the solution I offer below.

Many solutions to the free choice permission problem rely on pragmatic mechanisms for much of the heavy lifting, including Kamp 1978, Zimmermann 2000, Fox 2007, and others. The arguments that free choice implications are pragmatic, and more specifically are scalar implicatures, stem from discussions of indefinites in Kratzer and Shimoyama 2002, as developed by Alonso-Ovalle 2006 and Fox 2007. The main evidence that free choice implications may be scalar implicatures turns on the behavior of negated permission sentences (*You may not eat an apple or a pear*); I show how the analysis here can explain the behavior of such sentences in section 5.

In contrast to the pragmatic approaches, I will argue that the main free choice implications, including especially the implications from (1a) to (1b) and to (1c), are matters of entailment. To the extent that the analysis here is viable, it calls into question whether free choice implications are indeed implicatures. I discuss other non-implicature approaches (e.g., Aloni 2007) in section 6.2.

2. CLASSICAL LOGIC VERSUS LINEAR LOGIC

The account of free choice given below will depend on understanding the basics of Linear Logic at a fairly deep level. Since Linear Logic is unfamiliar to most semanticists, I will spend considerable time presenting Linear Logic here. If you already know Linear Logic, or if you prefer to see the application to natural language first, by all means skip to the next section.

2.1. Classical logic

I will only introduce the elements of classical logic that will be relevant for comparison with Linear Logic in the discussion below. This will include conjunction, disjunction, negation, and Weakening, but not, for example, quantification.

Formulas. There is a set of atomic formulas $a, b, c, ...$, and a set of variables over formulas $A, B, C, ...$. If A and B are formulas, then $\neg A$, the negation of A, is a formula; $A \wedge B$, the conjunction of A and B, is a formula; and $A \vee B$, the disjunction of A and B is a formula. In addition, $A \rightarrow B$ is defined as an abbreviation of $\neg A \vee B$.

Sequents. A sequent $A, B, ..., M \vdash N, O, ..., Z$ consists of two multisets of formulas joined by a turnstyle ('$\vdash$'). Sequents are interpreted as asserting that whenever all of the formulas in the leftmost multiset hold, then at least one of the formulas in the rightmost multiset must (also) hold.

Saying that a sequent contains multisets rather than lists of formulas means that the order in which formulas are written is immaterial. Thus A, B is a different multiset than A, A, B (since only the second one contains two instance of A), but A, B and B, A represent the same multiset. (Equivalently, we could define sequents using lists of formulas instead of multisets and adopt Exchange as a structural inference rule; Exchange allows formulas to be freely reordered.)

Capital Greek letters (Δ, Γ) schematize over (possibly empty) multisets of formulas. The turnstyle can occur in any position, and there can be more than one formula on the right hand side, so that the expression $\Delta \vdash A, B$, the expression $\Delta \vdash$, and the expression $\vdash \Delta$ are all legitimate sequents.

Negation. With the interpretation of a sequent just given, the following pair of inference rules should intuitively seem valid:

$$\frac{\Delta, A \vdash \Gamma}{\Delta \vdash \neg A, \Gamma}\neg_1 \qquad \frac{\Delta \vdash A, \Gamma}{\Delta, \neg A \vdash \Gamma}\neg_2$$

Beginning with $\neg_1$, the inference rule on the left: if Γ follows from the formulas in Δ along with A (this is what the sequent above the horizontal line expresses), then from Δ alone we can conclude that either some member of Γ is still true, or else A must be false (the sequent below the horizontal line). Similar reasoning applies for the inference rule on the right, $\neg_2$.

Proof. We now consider our first proof.

$$\dfrac{\dfrac{A \vdash A}{\vdash \neg A, A}\,\neg_1}{\neg\neg A \vdash A}\,\neg_2$$

A proof that a sequent is valid begins with trivial tautologies of the form $\Delta \vdash \Delta$. As long as each subsequent inference step instantiates a valid inference rule, the proof guarantees that the final sequent will also be valid. A sequent at the bottom of such a proof is called a theorem of the logic.

Reading from top to bottom, the first step of the proof here is an instantiation of the inference rule $\neg_1$. This step concludes that either A or its negation must be true (a version of the law of excluded middle); the second step (labeled $\neg_2$) proves that two adjacent negations cancel out (the law of double negation). (Proving that $A \vdash \neg\neg A$ is equally easy.)

Conjunction. In classical logic, the following inference is valid:

$$\dfrac{\Delta \vdash A \qquad \Delta \vdash B}{\Delta \vdash A \wedge B}\,\wedge$$

If the assumptions in Δ allow you to prove that A is true (i.e., if $\Delta \vdash A$), and the very same set of assumptions also allow you to prove that B is true, then you are certainly in a position to assert that the conjunction of A and B must be true (in symbols, $A \wedge B$).

Weakening. Of all the classical rules discussed here, only this one will have no counterpart in the Linear Logic system to follow.

$$\dfrac{\Delta \vdash \Gamma}{\Delta, A \vdash \Gamma}\,\text{Weakening}$$

If Γ follows from Δ, then Γ certainly still follows if A also happens to be true, no matter what A happens to express. The assumption A is gratuitous, but harmless. Weakening allows us to pick and choose among evidence as we focus on different parts of an argument.

$$\dfrac{\dfrac{A \vdash A}{A, B \vdash A}\,\text{Weakening} \qquad \dfrac{B \vdash B}{B, A \vdash B}\,\text{Weakening}}{A, B \vdash A \wedge B}\,\wedge$$

This proof shows how a single set of assumptions (here, the multiset of formulas A, B) can be used to prove two independent conclusions. Crucially, in this proof it is necessary to use Weakening in each subproof leading up to the $\wedge$ inference, in order to ignore whichever portion of the initial assumptions are not relevant for the subproof at hand.

Disjunction. For disjunction, we have a matched pair of inferences:

$$\frac{\Delta \vdash A}{\Delta \vdash A \lor B} \lor_1 \quad \frac{\Delta \vdash B}{\Delta \vdash A \lor B} \lor_2$$

If the assumptions in Δ allow you to prove that some proposition A is true, you can conclude that the (classical) disjunction of A and B is true. After all, if you know that Ann arrived, then you know that either Ann arrived or Bill arrived. The reason we need a pair of rules is that disjunction is symmetric, i.e., we are free to add the new disjunct either on the left or on the right.

The classical duality of conjunction and disjunction. The following equivalences hold:

(4) a. $\neg\neg A \equiv A$
 b. $\neg(A \land B) \equiv \neg A \lor \neg B$
 c. $\neg(A \lor B) \equiv \neg A \land \neg B$

The last two (DeMorgan's laws) express the logical interrelationship between disjunction and conjunction. These equivalences can be thought of as bi-directional inference rules. In any case, I will freely replace formulas with forms deemed equivalent by (4).

Implication as a form of disjunction. Recall that we defined classical implication $A \to B$ as $\neg A \lor B$. The inference rule that characterizes implication is Modus Ponens, which says that $A, A \to B \vdash B$ is valid. We can prove Modus Ponens as follows. The key aspect of the proof that is relevant for motivating Linear Logic is the role of Weakening.

$$\cfrac{\cfrac{\cfrac{\cfrac{A \vdash A}{A, \neg B \vdash A}\text{Weak} \quad \cfrac{\neg B \vdash \neg B}{\neg B, A \vdash \neg B}\text{Weak}}{\neg B, A \vdash A \land \neg B}\land}{A, \neg(A \land \neg B) \vdash \neg\neg B}\neg_1, \neg_2}{A, A \to B \vdash B}\equiv$$

As before, the $\land$ inference forces Weakening.

Wadler 1993 uses the following proof to emphasize the differences between classical logic and Linear Logic:

$$\cfrac{\cfrac{A \vdash A}{A, A \to B \vdash A}\text{Weak} \quad \cfrac{\text{[see previous proof]}}{A, A \to B \vdash B}}{A, A \to B \vdash A \land B}\land$$

The key thing is that Weakening allows us to make use of assumption A twice: once to justify the left conjunct of the conclusion, and once to support modus ponens in order to derive the right conjunct of the conclusion. In

Linear Logic, each assumption can be used exactly once, so, as we will see, this proof will not go through.

2.2. Linear Logic

In Linear Logic, Weakening is no longer allowed. (The classical structural rule of Contraction is also rejected, but Contraction does not play a role in the exposition here.)

Formulas. Once again there is a set of atomic formulas $a, b, c, \ldots$, and a set of variables over formulas $A, B, C, \ldots$. There are, however, new connectives. In fact, since none of the Linear Logic connectives mean what their classical counterparts mean, Linear Logic uses a completely distinct set of connective symbols, as follows: If A is a formula, then $A^{\perp}$, the linear negation of A, is a formula. For conjunction, we now have an additive conjunction '&', pronounced 'with', and a multiplicative conjunction '$\otimes$', pronounced 'times', so that if A and B are formulas, then $A \& B$ and $A \otimes B$ are also formulas. Likewise, for disjunction, we now have an additive disjunction '$\oplus$', pronounced 'plus', and a multiplicative disjunction '$\parr$', pronounced 'par', so that $A \oplus B$ and $A \parr B$ are formulas. (Note that many things in natural language semantics are called 'additive'. The Linear Logic notions of 'additive' and 'multiplicative' do not line up with any of them.) Linear implication, $A \multimap B$, where $\multimap$ is pronounced 'lollipop', is defined as an abbreviation for $A^{\perp} \parr B$.

Sequents. As before, though using linear formulas instead of classical formulas. Refining the semantic interpretation of sequents for Linear Logic is postponed till the general discussion of semantic models in section 7. In the meantime, sequents play the same role in inferences and proofs in Linear Logic as they do in classical logic.

Fragment of Linear Logic for the free choice permission problem. Here is the complete set of rules of Linear Logic that we will use in the discussion of the free choice permission problem, gathered together:

Fragment of Linear Logic for FCP

$$\frac{\Delta, A \vdash \Gamma}{\Delta \vdash A^{\perp}, \Gamma}\ \perp_1 \qquad \frac{\Delta \vdash A, \Gamma}{\Delta, A^{\perp} \vdash \Gamma}\ \perp_2$$

$$\frac{}{A \vdash A}\ \text{Axiom}$$

$$A \multimap B \equiv A^{\perp} \parr B$$
$$A^{\perp\perp} \equiv A$$

$$(A \,\&\, B)^{\perp} \equiv A^{\perp} \oplus B^{\perp} \qquad (A \otimes B)^{\perp} \equiv A^{\perp} \parr B^{\perp}$$
$$(A \oplus B)^{\perp} \equiv A^{\perp} \,\&\, B^{\perp} \qquad (A \parr B)^{\perp} \equiv A^{\perp} \otimes B^{\perp}$$

$$\frac{\Delta \vdash A \qquad \Delta \vdash B}{\Delta \vdash A \,\&\, B}\ \& \qquad \frac{\Delta \vdash A \qquad \Gamma \vdash B}{\Delta, \Gamma \vdash A \otimes B}\ \otimes$$

$$\frac{\Delta \vdash A}{\Delta \vdash A \oplus B}\ \oplus_1 \qquad \frac{\Delta \vdash B}{\Delta \vdash A \oplus B}\ \oplus_2 \qquad \frac{\Delta \vdash A, B}{\Delta \vdash A \parr B}\ \parr$$

Linear conjunction and disjunction. The rules for & and $\oplus$ (the so-called 'additive' connectives) look exactly like the classical rules for $\wedge$ and $\vee$, except for the substitution of & for $\wedge$ and of $\oplus$ for $\vee$. However, as a result of how they interact with the rest of the logic, the linear logic additives behave differently from their classical counterparts. For instance, the law of the excluded middle is a tautology for classical disjunction[2] (i.e., $\vdash \neg A \vee A$). In Linear Logic, the law of excluded middle is not valid for $\oplus$ (that is, $\nvdash A^{\perp} \oplus A$), though it is a tautology for multiplicative disjunction ($\vdash A^{\perp} \parr A$).
Linear negation. We have direct analogs to the classical rules for pushing a formula across the turnstyle, namely, $\perp_1$ and $\perp_2$. Since we now have two kinds of conjunctions and two kinds of disjunctions, there are more duality equivalences; however, each conjunction is still dual to a disjunction, and vice-versa.
Linear implication. Once again, we have defined implication in terms of disjunction. Now, interestingly, we can prove the linear version of Modus Ponens without using Weakening (which is a good thing, since Weakening

[2]In the presence of Contraction, which say that from $\Delta \vdash A, A$ infer $\Delta \vdash A$.

is not allowed in Linear Logic):

$$\dfrac{\dfrac{\dfrac{A \vdash A \qquad B^{\perp} \vdash B^{\perp}}{A, B^{\perp} \vdash A \otimes B^{\perp}}\otimes}{A, (A \otimes B^{\perp})^{\perp} \vdash B^{\perp\perp}}\perp_1, \perp_2}{A, A \multimap B \vdash B}\equiv$$

Because the inference rule for $\otimes$ splits up the resources into those used to prove A and those used to prove B, there is no need to ignore (locally) gratuitous assumptions via Weakening.

If we try to reproduce Wadler's proof from the previous section, we're out of luck:

$$\dfrac{?? \vdash A \qquad ?? \vdash B}{A, A \multimap B \vdash A \otimes B}\otimes$$

We could take some of the resources to the left of the turnstyle to prove A, and we could take some (actually, we would need all) of the resources to prove B, but no matter how we divide up the left-hand formulas, we'll fall short of proving one or the other of the conjuncts.

2.3. Choice

Since free choice permission is about making choices, what does Linear Logic have to say about choice?

The critical connectives will be the additive conjunction '&' and its (also additive) disjunctive dual, '$\oplus$'. The relevant inference rules are repeated here:

$$\dfrac{\Delta \vdash A \qquad \Delta \vdash B}{\Delta \vdash A \,\&\, B}\& \qquad\qquad \dfrac{\Delta \vdash A}{\Delta \vdash A \oplus B}\oplus_1 \quad \dfrac{\Delta \vdash B}{\Delta \vdash A \oplus B}\oplus_2$$

Imagine yourself in the role of the prover. Then the assumptions on the left of the turnstyle are what your environment gives you to work with, and the conclusion on the right of the turnstyle is what you return as the result of your labors (perhaps to be used as an assumption in a larger proof).

So here is what the & inference says: if receiving the resources in Δ allow you to provide A, and if the same set of resources allow you to provide B, then you can certainly offer to provide either A or B. Furthermore, since you are equally prepared to provide either alternative, you can leave the choice up to whoever might be interested in making use of the conclusion. Thus & conjoins two equally viable alternatives.

Though both alternatives are equally viable, the consumer is forced to choose between them. For instance, imagine that Δ contains a certain amount of sugar and a certain number of eggs. You offer "meringue & cake" for dessert, and you let your guest choose. Using the resources provided, you can construct either a meringue or else an angel food cake, but you don't have enough ingredients to cook both. (Note that in English, "meringue & cake" is pronounced "meringue **or** cake"—this is a point that we will return to in section 4.1.)

In the context of granting permission, the consumer is the entity to which permission has been granted: we shall see that (unembedded) & corresponds to free choice on the part of the entity given permission.

Dually, examining the $\oplus_1$ inference rule, if the resources in Δ allow you to provide A, then you can certainly offer to provide either $A \oplus B$—as long as you remain in control of which of the alternatives is chosen. You may only know how to make one dessert, perhaps. You can truthfully promise that dessert will either be meringue or else Baked Alaska, although you know in advance that it will have to be meringue. (Analogously for $\oplus_2$.)

In the context of granting permission, offering $A \oplus B$ does not give the grantee free choice.

In order to complete the picture of the dualities of & and $\oplus$, we must consider what happens on the other side of the turnstyle. Hopping across the turnstyle involves negation, which exchanges & for $\oplus$ (and vice versa).

$$\frac{A \vdash \Delta}{A \,\&\, B \vdash \Delta} \quad \frac{B \vdash \Delta}{A \,\&\, B \vdash \Delta} \qquad\qquad \frac{A \vdash \Delta \qquad B \vdash \Delta}{A \oplus B \vdash \Delta}$$

These rules follow from the official inference rules by applications of $\perp_1$ and $\perp_2$.

If A alone is enough to enable you to provide Δ, then if someone promises you $A \,\&\, B$, you can certainly commit to providing Δ: just select A when they give you your choice. (Similarly for the other rule introducing & on the left of the turnstyle.)

Finally, if having A is enough for you to offer Δ, and if having B is likewise enough for you to be able to offer Δ, then you're in a position to promise Δ even if all you can count on is $A \oplus B$. All you know is that you'll get either an A or a B, and that which one you get will be someone else's choice. However, since you are prepared to cope with either possibility, you can commit to providing Δ.

Bottom line: & and $\oplus$ are two perspectives on a single choice: & provides two equally legitimate alternatives, but forces an unconstrained (free) choice between them; $\oplus$ also provides two alternatives, but reserves the choice for whoever is providing the resource.

3. The Anderson/Kanger deontic reduction strategy

The Lokhorst 1997 analysis relies on using Linear Logic to implement a version of the so-called deontic reduction strategy for reasoning about obligation and permission. The strategy is attributed independently to Anderson and to Kanger (and is also associated with comments in Liebniz). Standard deontic logics introduce unary modalities representing obligation ($\Box$) and permission ($\Diamond$), and add axioms that characterize an appropriate set of entailments, usually including at least K and D, though there is considerable variation; see McNamara 2006 or Portner 2009 for an introduction to deontic logic. In contrast, the reduction strategy depends on a special proposition δ (pronounced "yay"), glossed as 'the good thing', or 'all things are as required'. More technically, Lokhorst 2006 shows that if we allow quantification over propositions, δ corresponds to the notion 'every proposition that is obligatory holds'.

Then A is obligatory iff $\delta \multimap A$: if A follows from the state where all things are as required, then A is required. Dually, a weak version of permission is often defined as $(\delta \multimap A^{\perp})^{\perp}$: if the negation of A is not obligatory, then A is in some sense not forbidden. However, there is a difference between weak permission, which is the absence of prohibition, and strong permission (i.e., a permissive norm), which is the assertion that some action is explicitly ok (see, e.g., Hansen et al. 2007). There is not much discussion of weak permission versus strong permission in the linguistics literature, but at least Asher and Bonevac 2005 conclude that free choice permission involves strong permission. In any case, I will assume that at least some natural language modals express strong permission, including *may*.

Lokhorst 1997 renders strong permission as $A \multimap \delta$. Portner 2009:60 comments on a similar formula to the effect that it is not easy to make sense out of this as a statement of permission. However, thinking in terms of linear implication rather than material implication helps. Viewing a linear implication as a function that turns objects of the antecedent type into objects of the consequent type, then a permissive norm such as $A \multimap \delta$ is a function that turns actions of type A into a situation in which all requirements have been met. Strong permission, then, amounts to providing a procedure for reconciling an action with the set of all requirements. In particular, note that such a procedure can only exist if A is at least consistent with all requirements. But this is a place where it becomes important to consider how strict implication differs from material implication: as Restall 2000:172,184 puts it, in order for $A \multimap \delta$, A and δ must "share some content". That is, what $A \multimap \delta$ asserts is that the only way to give a complete characterization of what is and is not required is by mentioning apple-eating.

The reason this strategy is called a reduction is that if we define obligation and (strong) permission as $A \multimap \delta$ and $\delta \multimap A$, we no longer need unary modalities $\Box$ or $\Diamond$—as long as the logic provides a 'strict' implication, where strict implication includes at least the relevant implication of Relevance Logic, or, as here, linear implication (though certainly not classical material implication). The idea is that strict implication guarantees that the antecedent has some critical role to play in making the consequent true: it must be "relevant", as in Relevant Logic, and the resource accounting of Linear Logic guarantees that the antecedent will be used (exactly once) in the process of producing the consequent. See McNamara 2006 and Lokhorst 2006 for results characterizing the sense in which the Anderson/Kanger reduction using Linear Logic includes all the theorems of standard deontic modal logics.

Adopting the Anderson/Kanger approach will be simpler from an expository point of view, since it enables us to talk about permission without extending the logic with unary logical connectives. However, it is not an innocent choice for the main empirical phenomenon under consideration here. As I will explain shortly, because linear implication is defined as $A \multimap B \equiv A^{\perp} \parr B$, the formula for which permission is granted (i.e., A) occurs in a downward-entailing position. This will be crucial in deriving the desired entailments. For all I know, however, it is possible that if a suitable notion of strong permission were defined in a standard deontic framework (i.e., one based on unary operators like $\Box$), similar entailments would go through.

4. Free choice permission

We can now suppose that *or* has among its meanings $\oplus$, so that *You may eat an apple or$_{\oplus}$ a pear* translates as $(a \oplus p) \multimap \delta$: the additive disjunction of a and p is explicitly permitted. Then the desired free-choice implication follows directly from simple linear reasoning. Generalizing slightly by using variables over formulas (A, B) instead of atomic formulas (a, p), we have:

$$
\dfrac{
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\dfrac{\vdash A, A^{\perp}}{\vdash A \oplus B, A^{\perp}}\,\oplus_1 \qquad \vdash \delta^{\perp}, \delta
}{\vdash (A \oplus B) \otimes \delta^{\perp}, A^{\perp}, \delta}\,\otimes
}{\vdash (A \oplus B) \otimes \delta^{\perp}, A^{\perp} \parr \delta}\,\parr
\qquad
\dfrac{
\dfrac{
\dfrac{\vdash B, B^{\perp}}{\vdash A \oplus B, B^{\perp}}\,\oplus_2 \qquad \vdash \delta^{\perp}, \delta
}{\vdash (A \oplus B) \otimes \delta^{\perp}, B^{\perp}, \delta}\,\otimes
}{\vdash (A \oplus B) \otimes \delta^{\perp}, B^{\perp} \parr \delta}\,\parr
}{
\dfrac{\vdash (A \oplus B) \otimes \delta^{\perp}, (A^{\perp} \parr \delta) \,\&\, (B^{\perp} \parr \delta)}{(A \oplus B) \multimap \delta \vdash (A \multimap \delta) \,\&\, (B \multimap \delta)}\,\perp_2, \equiv
}\,\&
$$

This theorem is noted in Lokhorst 1997:6.

What the speaker asserts when she utters *You may eat an apple or$_{\oplus}$ a pear* is that she is in possession of sufficient resources to prove either that eating an apple is permitted, or to prove that eating a pear is permitted. She is not asserting that she has enough resources to prove both, so if her utterance is to provide the justification for action, a choice must be made. However, since the resources allow proof of either alternative, the consumer is free to choose whichever of the alternatives he prefers. That is how the addressee can have permission to eat an apple, or else permission to eat a pear, but normally (and certainly not by virtue of the utterance of (1a)) does not have permission to eat two pieces of fruit.

This result depends on only two assumptions: that *or* can express additive disjunction in a resource-sensitive domain; and that it is reasonable to represent strong permission using the Anderson/Kanger deontic reduction. The assumption that *or* can express additive disjunction is essential, and is the heart of the explanation offered here. The deontic reduction is a standard approach to deontic logic motivated entirely independently of any concern with the free choice permission problem. Whether it can be replaced with a modal system more familiar to linguists remains for future work.

4.1. An ambiguity hypothesis: that *or* can express $\oplus$ or $\&$

Recall that (1a) (analyzed immediately above) and (3b), repeated here, are candidates for paraphrases of each other:

(5) a. You may eat an apple or a pear.
  b. You may eat an apple or you may eat a pear.

We have supposed that *or* can translation as $\oplus$. This should seem natural, since $\oplus$ is a type of disjunction. I would like to suggest that in addition, *or* can translate as &. We would immediately have an explanation for how (6a) could mean the same thing as (6b):

(6) a. You may eat an apple or$_\oplus$ a pear. $(A \oplus B) \multimap \delta$
  b. You may eat an apple or$_\&$ you may eat a pear. $(A \multimap \delta) \& (B \multimap \delta)$

We've already seen that (6a) entails (6b), since that is just the basic free choice implication; but in fact (6b) entails (6a) as well:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash (A \multimap \delta)^\perp, A \multimap \delta}{\vdash (A \multimap \delta)^\perp \oplus (B \multimap \delta)^\perp, A^\perp, \delta}\oplus \qquad \dfrac{\vdash (B \multimap \delta)^\perp, B \multimap \delta}{\vdash (A \multimap \delta)^\perp \oplus (B \multimap \delta)^\perp, B^\perp, \delta}\oplus}{\vdash (A \multimap \delta)^\perp \oplus (B \multimap \delta)^\perp, A^\perp \& B^\perp, \delta}\&}{\vdash (A \multimap \delta)^\perp \oplus (B \multimap \delta)^\perp, (A \oplus B)^\perp \parr \delta}\parr}{(A \multimap \delta) \& (B \multimap \delta) \vdash (A \oplus B) \multimap \delta}\perp_2$$

So the proposed translations of (6a) and (6b) are logically equivalent, which would explain why they can serve as paraphrases of each other.

Since & is a conjunction, *or*$_\&$ would be a truly conjunctive use of *or*! In fact, it has long been supposed that *or* might be ambiguous between a purely disjunctive use (corresponding here to $\oplus$) versus a meaning containing some conjunctive content, traditionally called "exclusive *or*". In the standard treatment, the exclusive meaning is described using classical connectives as $(a \vee b) \wedge \neg(a \wedge b)$, and it often assumed to be an implicature rather than an entailment.

Although & certainly does not mean the same thing as exclusive *or*, there is an exclusive flavor to &: as explained above, although both alternatives are equally viable, there are only enough resources to provide one of them, so a choice must be made.

The hypothesis, then, is that English *or* does not express disjunction, but rather, choice among alternatives. The two translations, $\oplus$ and &, differ only in whether the choice belongs to the producer of a resource ($\oplus$) or else the consumer of a resource (&). If so, then perhaps the reason people have so often supposed that *or* can have an exclusive interpretation is that this was the best simple approximation of & provided by classical logic.

Of course, if *or* is ambiguous between & and $\oplus$, we should also expect the properly disjunctive version $or_{\oplus}$ to occur in the non-embedded position, and in fact, this meaning is attested in the literature:

(7) You may eat an apple $\text{or}_{\oplus}$ you may eat a pear. $(A \multimap \delta) \oplus (B \multimap \delta)$

This sentence offers a single piece of fruit, but does not reveal which type of fruit is permitted. This interpretation is salient in a situation in which it would be appropriate to continue with *...but I don't know which*, or, somewhat more exotically, *...but I won't tell you which*, or perhaps *...see if you can guess which!*.

And to complete the paradigm, we would likewise expect $or_{\&}$ to occur in embedded position:

(8) You may eat an apple $\text{or}_{\&}$ a pear. $(A \mathbin{\&} B) \multimap \delta$

The translation in (7) entails the one in (8) (but not vice-versa; see section 7), so the meaning portrayed in (8) would likewise be appropriate in a *...but I don't know which* scenario.

4.2. Strengthening the antecedent

In addition to explaining how unembedded *or* can provide an exact paraphrase of embedded *or*, the hypothesis that *or* is ambiguous between the two additive connectives can also explain how permission talk behaves under strengthening of an antecedent.

Strengthening an antecedent is notoriously valid classically: from $A \to C$ conclude $(A \wedge B) \to C$. The fact that this is a theorem classically is one reason why material implication is generally considered inadequate as a rendering of natural language conditionals. For instance, it manifestly does not follow from *If I go to Spain, I will enjoy myself* that *If I go to Spain and get sick, I will enjoy myself*. But as often happens in Linear Logic, classical theorems are valid only for one sub-connective and not the other.

For instance, as mentioned above, the law of excluded middle is valid for ⅋ but not for $\oplus$ (i.e., $\nvdash A^\perp \oplus A$). Similarly, in Linear Logic strengthening of the antecedent is valid only for &, not $\otimes$:

$$\dfrac{\dfrac{A \multimap C \vdash A^\perp, C}{A \multimap C \vdash A^\perp \oplus B^\perp, C}\oplus}{A \multimap C \vdash (A \,\&\, B) \multimap C}\equiv, ⅋$$

Given that strengthening is valid for &, if English *and* could translate as &, we would expect (9a) to entail (9b):

(9) a. You may eat an apple.
b. You may eat an apple and a pear.

But this entailment is simply not available under any construal. However, since we now suspect that *or* can translate as &, we have

(10) a. You may eat an apple.
b. You may eat an apple or$_\&$ a pear.

Crucially, this entailment relation is only predicted to go through when (10b) receives an interpretation like (8), on which the addressee receives permission to eat exactly one kind of fruit. Gaining intuitive access to this entailment requires careful alignment of the pragmatic situation. Imagine that a parent tells her child (10a) in the presence of the babysitter. The afternoon passes, and it's finally time for a snack. The babysitter then utters (10b), continuing with *...but I don't remember which*. In that use, (10b) is true by virtue of (10a) being true, i.e., (10a) entails (10b), exactly as predicted by the validity of (the Linear Logic refinement of) strengthening of the antecedent.

4.3. Summary of the basic predictions

At this point, we have one clear reason to suspect that *and* cannot translate as &, since trying to strengthen an antecedent using *and* in (9) does not go through. We also have two reasons to suspect that *or* can translate as either $\oplus$ or &: the ambiguity hypothesis explains the synonymy of unembedded free-choice uses of *or* with embedded free-choice *or*, and it confirms strengthening predictions.

In a nutshell, the hypothesis is that *or* expresses choice, whether from the conjunctive perspective (consumer's choice) or the disjunctive perspective (producer's choice).

Assuming therefore that *or* is ambiguous between $\oplus$ and &, we have the following set of translations and their entailments.

	SENTENCE	TRANSLATION	ENTAILS
(11) a.	You may eat an apple.	$A \multimap \delta$	a, e, f
b.	You may eat a pear.	$B \multimap \delta$	b, e, f
c.	You may eat an apple or$_\oplus$ a pear.	$(A \oplus B) \multimap \delta$	a,b, c, d, e, f
d.	You may eat an apple or$_\&$ you may eat a pear.	$(A \multimap \delta) \& (B \multimap \delta)$	a, b, c, d, e, f
e.	You may eat an apple or$_\&$ a pear.	$(A \& B) \multimap \delta$	e
f.	You may eat an apple or$_\oplus$ you may eat a pear.	$(A \multimap \delta) \oplus (B \multimap \delta)$	e, f

The entailments from (a) and (b) to (e) is strengthening; (e) and (f) have meanings suitable in *...but I don't know which* situations.

It is worth emphasizing that all of these entailments are purely semantic, without requiring any silent (pragmatically-triggered) type shifting operators (as in, e.g., Fox 2007), or other pragmatic enrichment.

5. PROHIBITION

The behavior of permission under negation plays an important role in recent discussions. As mentioned above, Alonzo-Ovalle 2002 and Fox 2007 argue that the fact that free-choice implications seem to disappear under negation shows that free choice implications in the positive case are likely to be implicatures. Since I am claiming that the relevant free choice implications are entailments (especially the one analyzed at the beginning of section 4), it is important to carefully examine negated cases.

Whatever is not permitted is forbidden: just as in English, Lokhorst renders (strong) prohibition as negated (strong) permission. Thus if $(A \multimap \delta)^\perp$, then A is prohibited. (It is a well-known property of English that *may not* is always construed with negation taking scope over *may*.)

(12) a. You may not eat this apple or this pear.
b. You may not eat this apple.
c. You may not eat this pear.

The main fact to be explained is that (12a) entails (12b) and (12c). Unlike the positive free choice example discussed above, however, we can usually infer that (12b) and (12c) hold simultaneously (multiplicative conjunction). That is, under its most natural interpretation (12a) simultaneously forbids both apple-eating and pear-eating. Apparently, permission is a scarce resource, but prohibition is all too abundant. I will call this construal of (12a) the double-prohibition reading, and I will suggest that it arises as a standard Gricean implicature.

5.1. Double prohibition

Given our hypothesis that *or* is ambiguous, we have two possibilities to consider.

(13) a. You may not eat this apple $or_{\oplus}$ this pear.
b. You may not eat this apple $or_{\oplus}$ you may not eat this pear.

$$((A \oplus B) \multimap \delta)^{\perp} \vdash (A \multimap \delta)^{\perp} \oplus (B \multimap \delta)^{\perp}$$

The second interpretation involves $or_{\&}$, and is discussed below in section 5.2.

The translation of (13a) entails the translation of (13b), so we predict that (13a) ought to have an interpretation on which it guarantees that (13b) is true. Such an interpretation is widely attested in the literature, and usually is described as favoring the continuation ... *but I don't know which*. I'll call this the ignorance reading.

Note, by the way, if a forgetful babysitter utters (13) to her charge, if the child behaves rationally, he will not eat either piece of fruit, since he can't be sure which action is safe—exactly the same behavior as if both actions had been explicitly forbidden.

As a next step towards deriving the double-prohibition interpretation, consider a situation in which the speaker is not ignorant. Exactly one of the alternatives is prohibited, and this time the speaker knows which one it is. For the sake of concreteness, let's say that apple-eating is forbidden, but pear eating is fine. If the speaker were being fully cooperative, then she would normally choose to simply say (12b), and certainly would not choose to say (12a). In Gricean terms, adding a superfluous disjunct would violate the maxim of Manner.

There are nevertheless situations in which this kind of uncooperative statement might be used. For instance, if a father tells an older sister the rules ("apples forbidden, pears ok"), she might later uncooperatively tell her younger brother

(14) You may not eat this apple or this pear ... but I won't tell you which.

Once again, the rational course of action on the part of the younger sibling will be to refrain from eating either piece of fruit. Presumably this is exactly the outcome the unkind sister is aiming for. (I'm indebted to Sven Lauer for this scenario; see also Simons 2005:273n.4.)

In both the ignorance scenario and the uncooperative scenario, at least one of the disjuncts is guaranteed, but the choice of which fruit is prohibited is reserved for whoever is providing the (incomplete) information. The receiver of the prohibition must plan for the worst, and therefore can't safely commit to either alternative.

Finally, imagine that the speaker is neither ignorant nor uncooperative. She may be an expert (perhaps she just received full instructions from the parents) or she may be herself the source from which permission flows;

in any case, she is fully opinionated about what is forbidden. Crucially, although (13) guarantees only one disjunct, it is consistent with situations in which both disjuncts hold. As argued above, if exactly one disjunct held, the speaker would simply have said so. She certainly should not use the disjunctive statement. We can deduce, therefore, that both disjuncts must hold.

There is one more step to complete the Gricean explanation. If the speaker intends to convey double prohibition, why not use *and*?

(15) You may not eat an apple and a pear.

Although this sentence may have the desired double-prohibition reading, it also has a reading on which it prohibits (only) complex events that involve eating both an apple and a pear. Uttering (15), then, leaves in play the possibility that eating a single piece of fruit may be permitted. The speaker uses a weak form in (13) to express a stronger meaning in order to avoid misinterpretation.

Thus the assumption that the speaker is opinionated and cooperative derives the implicature that both disjuncts are prohibited via ordinary Gricean reasoning, without the need to stipulate any special uniformity or distributivity axioms (as in Alonzo-Ovalle 2002) or Zimmermann's 2000:286 Authority Principle.

5.2. Free choice prohibition?

If *or* is ambiguous, we have a second option to consider in which *or* expresses additive conjunction:

(16) a. You may not eat this apple $\text{or}_{\&}$ this pear.
b. You may not eat this apple $\text{or}_{\&}$ you may not eat this pear.

$$((A \,\&\, B) \multimap \delta)^{\perp} \vdash (A \multimap \delta)^{\perp} \,\&\, (B \multimap \delta)^{\perp}$$

If a speaker intends to convey the thought illustrated in (16), the addresses is forced to choose between the two prohibitions.

If such a reading exists, it is difficult to detect. We can try to draw it out with a carefully constructed scenario.

(17) a. You must eat an apple or a pear.
b. You may not eat more than one kind of fruit.
c. Therefore, you may not eat an apple or a pear (your choice!).

The net effect of the rules will be to forbid either eating of an apple (once a pear has been consumed) or else eating of a pear (once an apple has been consumed). Since the addressee has free choice of which kind of fruit to eat, they in effect have free choice of which kind of fruit will be forbidden.

The intended interpretation may come through more clearly with the unembedded coordination (*Therefore you may not eat an apple or you may not eat a pear*).

6. Comparisons with other accounts

6.1. Implicature accounts

A number of authors have suggested that free choice implications can sometimes be derived as a special kind of scalar implicature. For instance, Fox 2007 reasons as follows: if a speaker utters a disjunction when she could have made a stronger statement, this could naturally lead to a Quantity implicature that she did not have sufficient evidence to assert the stronger statement. If those ignorance implicatures are implausible, as when the speaker is describing permissions in a situation in which their judgment is authoritative, the implausibility can trigger a repair strategy under which the disjunction is pragmatically enriched by the application of a predicate `exh` (for "exhaustive"). For instance, if an authoritative speaker says *You may eat an apple or a pear*, it may be implausible that she doesn't know whether you may eat an apple, or whether you may eat a pear. Therefore the statement $\Diamond(\mathsf{A} \vee \mathsf{P})$ can be strengthened to an exhaustive meaning equivalent to the proposition $\Diamond\mathsf{A} \wedge \Diamond\mathsf{P} \wedge \neg(\Diamond(\mathsf{A} \wedge \mathsf{P}))$. This asserts that you may have an apple, and you may have a pear, but you may not both have an apple and a pear.

I will discuss three potential problems with the exhaustivity implicature approach. The first problem is that the free-choice reading can survive even when ignorance is asserted:

(18) I don't know whether you may have an apple or a pear.

Since exhaustivity is triggered by contexts that are incompatible with ignorance, (18) should only have a reading on which it means 'I don't know whether you may have an apple or whether you may have a pear'. But (18) robustly also has a free-choice reading on which it means 'I don't know whether you may eat a piece of fruit, where the fruit is your choice between an apple or a pear'.

The second problem is that if free choice implications were implicatures, we should expect them to be cancellable:

(19) You may eat an apple or a pear, although in fact you may not eat an apple.

Probably (19) does have an additive disjunction reading on which it is at least logically consistent. If this were the basic semantic meaning of (19), then we would expect it to emerge whenever the free-choice implication is cancelled. The puzzling thing is that if we assume the speaker is opinionated, (19) gives a strong impression of contradiction rather than of a cancelled implicature.

Interestingly, Chemla (ms) proposes a pragmatic principle that he calls symmetry, which says that the epistemic attitude of the speaker must be uniform across disjuncts. Symmetry correctly predicts that (19) should be infelicitous, since it implies that the speaker holds a different attitude towards one disjunct than towards the other. However, symmetry alone cannot explain why (19) should sound contradictory.

One possibility is that performativity is interfering. Portner (ms) suggests that performative uses (see section 8.2 below) force, or at least strongly promote, a free choice interpretation. If so, then what (19) shows is that at least when an utterance is performative, free choice implications cannot be cancelled.

The third problem is that, as Fox himself notes, the proposed implicatures for the free-choice reading may not match intuitions about the meanings of the sentences in question. Fox's `exh`-enhanced truth conditions assert that eating an apple is permitted, and eating a pear is permitted, but eating an apple and a pear is forbidden. But as Simons 2005 notes, free choice is compatible with joint permission, it merely fails to guarantee joint permission. For instance,

(20) [You may eat an apple, and you may also eat a pear, so]
You may (certainly) eat an apple or a pear.

On Fox's account, (20) should be contradictory on a free-choice reading of the final clause. However, although (20) may be mildly redundant, there is no hint of contradiction.

As explained above, I have suggested that *You may eat an apple or a pear* entails that you may eat an apple, and that you may eat a pear, but refrains from granting explicit permission to eat both an apple and a pear. Having two pieces of fruit is not forbidden, but neither is it guaranteed. Classical logic is not sufficiently expressive to encode the meaning I have in mind, at least not in any simple way. But we have seen above how to express it in Linear Logic.

Chemla 2009 summarizes the current state of the debate on the status of free choice implications viewed as implicatures, and provides empirical evidence against a certain class of pragmatic analyses that rely on calculating implicatures in embedded contexts.

6.2. Alternative set semantics

Zimmermann 2000 proposes that disjunction contributes a set of exhaustive epistemic alternatives, so that *You may eat an apple or you may eat a pear* expresses the claim that it is possible that you may eat an apple and it is possible that you may eat a pear. Novel pragmatic principles (notably his Authority Principle) strengthen this conjunction into an assertion that you may eat an apple and you may eat a pear.

Geurts 2005 elaborates on Zimmermann's analysis, arguing that disjunctive alternatives should not always be epistemic. Rather, disjunction "fuses" with nearby modal operators, so that *You may eat an apple or a pear* means that you may eat an apple and you may eat a pear without needing to invoke any special pragmatic principle.

Neither Zimmermann's nor Geurts' analyses explain why the free-choice *or* differs from an overt *and* (i.e., *You may eat an apple and you may eat a pear*) in failing to guarantee that two pieces of fruit may be eaten. In addition, as Geurts 2005:406 briefly discusses, it is not clear how either analysis accounts for negated free choice (discussed above in section 5).

The idea that disjunction introduces a set of alternatives has been implemented in a variety of different ways. Van Rooij 2008:309 gives an analysis on which alternatives are built into the definition of a minimal extension of a world. Then a world in which you eat only an apple might qualify as a minimal extension of the world we are in, but not a world in which you eat both an apple and a pear. In order for the account to deliver free choice implications, it is necessary for the propositions expressed by a disjunction to always be among those used for articulating minimal extensions, though this requirement is not guaranteed by the formal analysis.

Building in part on Zimmermann's work, Kratzer and Shimoyama 2002 propose that indefinites contribute a set of alternatives, one for each way of resolving the indefinite. This requires in turn a profound modification to the compositional semantics, since it is necessary to allow for composition with sets of meanings instead of single meanings. This is done pointwise ("Hamblin semantics"), so that an embedded indefinite can give rise to a set of alternatives at higher compositional levels (see Shan 2004 for insights into the complexities of pointwise composition). Alonzo-Ovalle 2006 extends this strategy from indefinites to disjunction, explicitly addressing the free choice problem.

Aloni 2007 also proposes that disjunction projects alternatives, although she does not use a Hamblin-style approach. Rather, she bases her special composition on the dynamic semantics of, e.g., Dekker 2002, supplemented with structured propositions.

In these approaches, free choice effects arise when certain operators explicitly manipulate alternative sets. For instance, Aloni stipulates that *may*Φ is true (where Φ is a set of alternatives) just in case the ordinary meaning of *may* is true of each alternative. Thus *You may eat an apple or a pear* involves applying *may* to the set of alternatives corresponding to the addressee eating an apple and the addressee eating a pear. The sentence will be true, then, just in case *You may eat an apple* is true and *You may eat a pear* is true.

The account here resembles Aloni's alternatives account in three important respects.

First, free choice implications are entailments rather than implicatures. As we saw in section 6.1, the fact that free choice implications do not always seem to be cancellable argues in favor of theories on which they are treated as entailments.

Second, because alternative-taking *may* requires that ordinary *may* must be true of every alternative, it is a downward-entailing operator with respect to the disjunction that gives rise to the alternatives. Aloni points out that this explains why (so-called free choice) *any* is licensed (e.g., *You may eat anything*), and since the antecedent of linear implication is likewise a downward-entailing position (as noted above), the same explanation carries over here.

Third, Aloni also postulates that *or* is ambiguous. On Aloni's account, *or* either (in effect) contributes two alternatives, giving rise to the free choice interpretation, or else it contributes a single alternative expressing an ordinary (classical) disjunction, giving rise to the *...but I don't know which* interpretation.

One important difference between the approach here and Aloni's analysis is the integration with the larger compositional system. Aloni's approach (likewise Hamblin semantics) in effect creates unbounded dependencies in the semantics: multiple-alternative *or* introduces alternatives, which the compositional system must track, until an alternative-aware operator collapses the alternatives back into to a single proposition. The account here arises from a refinement of the standard denotations, in the sense that Linear Logic is weaker (i.e., more expressive) than classical logic. The compositional system remains entirely undisturbed.

A second important difference concerns the explanation for the behavior of unembedded *or*.

(21) a. You may eat an apple or you may eat a pear.
b. You ate an apple or you ate a pear.

It is not clear how Aloni would account for the free choice reading of (21a). Because *or* is not under the scope of a modal, the operator that distributes over the alternatives would have to be some root-level or discourse-level operator. This may be viable (see discussions in Kratzer and Shimoyama 2002 and Alonso-Ovalle 2006); however, one challenge for such an operator is that it would have to somehow fail to apply in the ordinary (non-modal) case, since *You ate an apple or you ate a pear* does not have a free choice reading.

Like Geurts 2005, Simons 2005 emphasizes the importance of explaining the synonymy of embedded free choice *or* (*You may eat an apple or a pear*) and unembedded free choice *or* (*You may eat an apple or you may eat a pear*). She proposes an across-the-board LF movement operation on which the sentence with unembedded *or* is predicted to be logically equivalent to *You may* [*eat an apple or eat a pear*]. On the account here, the equivalence follows directly from assuming that *or* can express &.

7. SEMANTICS

The discussion so far has been entirely in terms of inference and proofs. It is unusual these days, though not unheard of, to express the meanings of natural language expressions using proof theory rather than a model theory. In any case, I will say something here about the semantics of the expressions in question.

There are a number of semantic approaches to Linear Logic. Girard's 1987 original semantics for linear logic involved quantales, which would not be especially helpful here. There are other approaches, however, that have tantalizing associations with the granting and denying of permission. I will mention three here. First, Petri nets describe the movement of tokens through a network. Lokhorst 1997 uses Petri nets as models of his Linear Logic treatment of deontic reasoning. (Think of the tokens as lumps of permission moving from one location to another.) Second, in game semantics a Proponent and an Opponent take turns making choices, and I have argued that tracking choice is central to understanding permission talk. See, e.g., Accorsi and van Benthem 1999 for a discussion of game semantics for Linear Logic. Third, there are computational models of Linear Logic that make an explicit connection between the additives and choice. For example, Abramsky's 1993 computational semantics for intuitionistic Linear Logic interprets $A \otimes B$ as an ordered pair $\langle A, B \rangle$ both of whose elements will be used in further computation (eager evaluation); $A \,\&\, B$, on the other hand, denotes an ordered pair only one of whose elements will ever be used (lazy evaluation), and of course $A \oplus B$ delivers a projection function that chooses one or the other of the elements in a & pair. Unfortunately for our purposes here, intuitionistic Linear Logic does not provide the full involutive negation required for the discussion of negated permission sentences, and Abramsky's computational interpretation of classical Linear Logic involves parallel distributed processing, which would take us too far afield.[3]

[3]Though it is intriguing to think that the meaning of some natural language expressions might be appropriately modeled by a distributed process. Perhaps some permission sentences denote programs which the recipient can execute in various environments in order to produce whichever certificate of permission is required. Then a free choice permission

Most reassuringly familiar for linguists, there is a perfectly kosher Kripke semantics provided in Allwein and Dunn 1993. As usual, propositions are modeled as sets of possible worlds. Not as usual, formulas are modeled as *pairs* of sets of worlds: the set of worlds where the formula is known to hold, and the set of worlds in which the formula is know to not hold. If a world is in neither set, then the formula may hold there or not, without prejudice. Conjunction and disjunction are not simple intersection and union, but rather make use of information from both the positive and negative extension of the formula. The Allwein and Dunn semantics is the official semantics of the analysis here, although there is not space to repeat the full details.

However, I will follow the lead of Restall 2000:170 and offer here a simpler approximation. One reason it is only an approximation is that it validates distributivity:

$$\frac{A \,\&\, (B \oplus C)}{\overline{(A \,\&\, B) \oplus (A \,\&\, C)}}$$

In the context of permission talk, this bi-directional inference rule would guarantee that the fact that you may eat an apple and either a banana or a pear is equivalent to the fact that you may either eat an apple and a banana, or eat an apple and a pear. Whether or not this inference is appropriate for modeling natural language, it is not required for any of the arguments made above, and therefore I will not take a stance here on whether it should be adopted or not (see Restall 2000 chapter 12 for a discussion of distributivity). If you favor distributivity, then the semantics given here will be that much closer to adequate[4]; but even if you reject distributivity, the semantics will nevertheless give a faithful impression of the difference between the semantics of Linear Logic and classical logic.

Classical formulas are bivalent, either true or false. They partition any set of worlds into two cells: the set of worlds that make the proposition true, and those that make it false. Linear Logic tracks not only truth, but falsehood: a formula can be certainly true (T), or not; and a formula can independently be certainly false (F) or not. This means that an evaluation point (a world) can fall into four bins: certainly true, certainly false, certainly true and also certainly false, and neither certainly true nor certainly false. These categories amount to four distinct truth values. Call these values `t`, `f`, `b`, and `n`, respectively (for true, false, both, and neither).

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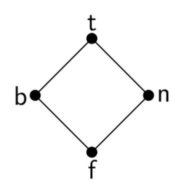
sentence denotes a program whose execution is blocked until it receives an external choice (a selection of which alternative to deploy).

[4]Though it would still not fully characterize the logic, since it validates the non-theorem $A \,\&\, B, A^{\perp} \vdash A \otimes B$.

We can easily give an epistemic interpretation to these categories. If a formula has the truth value n at a world, that can mean that the formula may be true, and may be false, we just don't know. In the context of granting permission, however, we can give a different interpretation. If it is within our power to determine what is permitted and what is not, a formula can be n if we simply haven't decided yet which actions are permissible. And if we are describing what is permitted and what is not based on some body of rules, it can easily happen that there are actions for which the rules do not determine permissibility.

Now, a world in which some formula is both certainly true and certainly false is a contradictory world. In the epistemic interpretation, it is difficult to imagine what such a world would be like. In the deontic interpretation, it is somewhat easier to imagine a world in which someone has given you permission to eat an apple and has also forbidden you to eat an apple. Nevertheless, many deontic logics explicitly outlaw contradictory sets of obligations. But whether or not your ontology allows contradictory worlds (that is, whether or not any evaluation points fall into the bin labelled b), we certainly can have formulas, both atomic and complex, that have b as a value. For instance, we can add to our logic an atomic formula $\bot$ whose value is b.

We can arrange the four truth values into the following Hasse diagram depicting a (distributive) lattice:



In this lattice, elements at the bottom of a line entail elements at the top of a line. Since entailment is reflexive and transitive, f entails all four values; b and n entail only themselves and t; and t entails only itself. Note that the element that models contradictions (namely, b) is not the bottom of the lattice.

The additive connectives & and $\oplus$ correspond to meet and join on the lattice:

&	t	b	n	f
t	t	b	n	f
b	b	b	f	f
n	n	f	n	f
f	f	f	f	f

$\oplus$	t	b	n	f
t	t	t	t	t
b	t	b	t	b
n	t	t	n	n
f	t	b	n	f

For instance, if a formula A has the value t, and B has the value n, then since the greatest lower bound for t and n in the lattice is n, the truth value for $A \,\&\, B$ will therefore be n.

The remaining connectives have values as follows:

⊗	t	b	n	f
t	t	t	n	f
b	t	b	n	f
n	n	n	f	f
f	f	f	f	f

⅋	t	b	n	f
t	t	t	t	t
b	t	b	n	f
n	t	n	t	n
f	t	f	n	f

⊸	t	b	n	f
t	t	f	n	f
b	t	b	n	f
n	t	n	t	n
f	t	t	t	t

The negation of a formula A is equivalent to $A \multimap \bot$:

¬	t	b	n	f
	f	b	n	t

At this point, we must refine the interpretation of sequents in two respects. First, in classical logic, we interpret $\Delta \vdash \Gamma$ as asserting that the conjunction of the formulas in Δ guarantees the disjunction of the formulas in Γ. Thus the comma that we use to write classical multisets of formulas means conjunction on the left of the turnstyle, and disjunction on the right, so that $A, B \vdash C, D$ is equivalent to $A \wedge B \vdash C \vee D$. In Linear Logic, we have two sorts of conjunction and disjunction, so we have to be more specific. For a linear sequent, the multiplicative conjunction of the formulas on the left guarantees the multiplicative disjunction of the formulas on the right, so that the linear sequent $A, B \vdash C, D$ is equivalent to $A \otimes B \vdash C ⅋ D$.

Second, because we have four semantic values rather than two, we must generalize our conception of what a sequent guarantees. Instead of saying that the truth of the assumptions guarantees the truth of the conclusion, we must say instead that the truth of the conclusion must be at least as great as the truth of the assumptions, where "at least as great" refers to the ordering imposed by the lattice. So if $\Delta \vdash \Gamma$, then for every possible way of assigning values to the atomic formulas in Δ and Γ, the value of the multiplicative conjunction of Δ must entail the value of the multiplicative disjunction of Γ.

To illustrate how these tables provide a model of the logic, recall that we have the following theorems and non-theorem:

(22) a. $(A \multimap \delta) \,\&\, (B \multimap \delta) \vdash (A \oplus B) \multimap \delta$ $\quad f \leq f$

b. $(A \oplus B) \multimap \delta \vdash (A \multimap \delta) \,\&\, (B \multimap \delta)$ $\quad f \leq f$

c. $(A \multimap \delta) \oplus (B \multimap \delta) \vdash (A \,\&\, B) \multimap \delta$ $\quad n \leq t$

d. $(A \,\&\, B) \multimap \delta \nvdash (A \multimap \delta) \oplus (B \multimap \delta)$ $\quad t \not\leq n$

If we assign $\mathtt{A}, \mathtt{B}$ and δ the values $\mathtt{b}$, $\mathtt{n}$, and $\mathtt{f}$, respectively, then the sequents evaluate to the values displayed in the righthand column, as you can check by inspecting the truth tables.

What can this semantics tell us about the meaning of the additive connectives? Since & is interpreted as meet in the value lattice, we immediately have that the value of $\mathtt{A} \,\&\, \mathtt{B}$ entails the value of $\mathtt{A}$, and that $\mathtt{A} \,\&\, \mathtt{B}$ likewise entails the value of $\mathtt{B}$. In other words, $\mathtt{A} \,\&\, \mathtt{B}$ will never be any more true than the least true of its conjuncts. These semantics faithfully correspond to the inference rule for &.

The explanation for the free choice phenomenon discussed above relies on the fact that & forces a choice between two equally viable alternatives. To see this aspect of the meaning of & from a semantic perspective, consider the following theorem and non-theorem:

(23) a. $\mathtt{A} \otimes \mathtt{B}, \mathtt{A} \multimap (\mathtt{B} \multimap \delta) \vdash \delta$ $\qquad \mathtt{f} \leq \mathtt{f}$
b. $\mathtt{A} \,\&\, \mathtt{B}, \mathtt{A} \multimap (\mathtt{B} \multimap \delta) \not\vdash \delta$ $\qquad \mathtt{n} \not\leq \mathtt{f}$

Multiplicative conjunction makes both of its conjuncts simultaneously available, so if we have $\mathtt{A} \otimes \mathtt{B}$, we can first use $\mathtt{A}$ to conclude that $\mathtt{B} \multimap \delta$, then go on to use $\mathtt{B}$ to conclude that δ. But in (23b), if we use up $\mathtt{A}$ to conclude $\mathtt{B} \multimap \delta$, we can no longer count on having $\mathtt{B}$ to complete the process, so we can't guarantee δ. Thus if we assign $\mathtt{A}, \mathtt{B}$ and δ the values $\mathtt{n}$, $\mathtt{n}$, and $\mathtt{f}$, respectively, the sequents evaluate as displayed in the rightmost column.

8. Some remaining issues

8.1. Epistemic and other potential free choice effects

It is widely assumed that whatever explains free choice implications for deontic modals is the same thing that explains the similar behavior of epistemic modals:

(24) a. John might be in Aarhaus or$_{\oplus}$ in Boston.
b. John might be in Aarhaus.
c. John might be in Boston.
d. John might be in Aarhaus or$_{\&}$ John might be in Boston.

The simplest way to extend the account here to epistemic cases would be to add to our logic a new atomic formula ϵ, which is true just in case everything that is epistemically known holds. Then *You might be in Aarhaus* would translate as $\mathtt{A} \multimap \epsilon$, and so on. To see the predictions in the epistemic case in more detail, just read the sentences in the chart in (11) giving *may* its epistemic rather than deontic sense.

Despite the similarity of the formal strategy proposed here, it is important to keep track of what the logic claims to be modeling. Classical logic promises to preserve truth: if the assumptions are true, the conclusion will be true. That is why it is legitimate to duplicate and discard assumptions. Linear Logic promises to preserve resources: whatever resources the assumptions provide, that is exactly what resources will appear in the conclusion. In the deontic case, the critical resource is permission: if the assumptions provide enough permission to eat exactly one piece of fruit, then the conclusion will provide the same amount of permission. In the epistemic case, the critical resource is epistemic commitment: whatever commitments are made by the assumptions, the conclusion will make exactly the same commitments.

As far as I know, reduction strategies are not usually proposed for other modalities besides the deontic. Certainly the logic of epistemic reasoning is significantly different from deontic reasoning. For instance, it is generally considered desirable for an epistemic logic to guarantee that if you know that A is true, then A is true ($\Box A \vdash A$). But deontically, you would not want to conclude from the fact that A is obligatory that A must be true, since obligations are all too often not fulfilled. More relevantly, there are empirical dis-analogies between the free choice behavior of deontic uses of modals versus epistemic modals. For instance, Kamp 1978, Zimmermann 2000, and Aloni 2007 note that it is significantly more difficult to construe epistemic modals as having a *...but I don't know which* interpretation (though it is still possible—see especially Simons 2005:274). In addition, here is a contrast between deontic and epistemic modality pointed out to me by Cleo Condoravdi (personal communication):

(25) a. You must not eat an apple or a pear.
b. You may not eat an apple or a pear.

(26) a. John must not be in Aarhaus or in Boston.
b. John may not be in Aarhaus or in Boston.

The deontic pair have a reading on which they are closely synonymous. The epistemic pair, however, can never be paraphrases of each other.

I'm not aware of any reason why a reduction strategy could not be part of a more complete analysis of epistemic modality; nevertheless, it would be prudent to be cautious about assuming that any deontic analysis should automatically extend to epistemic cases.

Fox 2007 suggests that free choice effects can be discerned in non-modal contexts that involve existential quantifiers.

(27) There's beer in the fridge or in the cooler out back.

Especially when (27) is heard as an implicit permissive, (27) entails both that there is beer in the fridge and that there is beer in the cooler out back. In our terms, this is $or_{\&}$: both alternatives are guaranteed to be true, and the consumer of the information has their free choice of which one is relevant for forming a plan of action.

Klinedinst 2007 suggests that free choice effects are present with some existential quantifiers, but only when the quantificational DP is plural:

(28) a. Some passengers got sick or had difficulty breathing.
b. A passenger got sick or had difficulty breathing.

In (28a), there is a reading on which some passengers got sick, and some had difficulty breathing. On such a reading, at least some of the passengers must have gotten sick, and at least some of the passengers must have had difficulty breathing. But in (28b), there is no guarantee that both of the properties must be instantiated.

Chemla 2009 discusses experiments that provide empirical data on the implications of free-choice sentences involving quantifiers.

In addition, Kratzer and Shimoyama, as well as Alonzo-Ovalle 2006, argue that free choice is relevant for the interpretation of indefinites, and Aloni 2007 argues that it is relevant for the interpretation of alternative questions and imperatives.

8.2. Performativity

Kamp 1978 draws a distinction between granting permission versus describing permission, where granting permission is a performative action. When a parent says *You may eat an apple or a pear* in the right circumstances, fruit-eating options may come into being that were not present before the utterance. But when a sibling comments *Apparently, you may eat an apple or a pear*, they are merely describing the current situation, and no new options come into being. Van Rooij 2008 and Portner (ms) develop a dynamic semantics for permission on which a permission sentence performatively changes the set of what is allowed.

One of the main arguments that performativity is important relies on correlations between performative uses and free choice uses. Certainly descriptive uses (such the sibling's comment) can have a free choice interpretation or not. Performatives, however, strongly prefer a free choice interpretation. Yet it may still be possible for a performative to have a non-free choice interpretation:

(29) You may pillage city X or city Y. But first take counsel with my secretary.

Kamp 1973:67 says of this example that "[t]he second part of this statement makes it clear that the vassal should not infer from the first part that he may make his own choice of city. Which one he may loot ultimately depends on the secretary's advice, the tenour of which—we may assume—is at this point unknown to king and vassal alike." To be sure, nothing specific has been permitted, and the vassal cannot form a complete plan of action. If we conceive of a performative as something that enlarges what an agent may safely do, we might therefore suppose that (29) is a merely descriptive use after all. Yet something must have been permitted: where does the disjunctive permission that the sentence describes come from, if not from the performance of (29)?

As far as the current paper is concerned, it is enough for permission sentences to characterize what is allowed. Then whether an utterance expands the sphere of permissibility depends on the interaction of the truth conditions with the normal range of factors that influence how a discourse participant decides to react to an utterance. Whether this minimalist strategy is viable, or whether it will ultimately be necessary to provide a special role for performativity remains to be seen.

8.3. Disjunctive *and*?

If *or* expresses additive choice, either & or $\oplus$, then we should look to see whether *and* expresses the dual pair of concepts: multiplicative simultaneity, either $\otimes$ or $⅋$. The $\otimes$ part seems obvious enough, but $⅋$ is less obvious. Since $\mathsf{A} ⅋ \mathsf{B}$ is equivalent to $\mathsf{A}^{\perp} \multimap \mathsf{B}$, the prediction is that there should be use of *and* in which it expresses something like the reverse of *unless* (since B unless A means roughly $\mathsf{A}^{\perp} \multimap \mathsf{B}$).

(30) a. Don't strike this match and it will light.
b. Strike this match and it will light.
c. If you strike this match, it will light.

More specifically, the prediction is that (30a) should be able to mean something like (30c). It doesn't, of course; however, (30b) (surprisingly) does. Following Russell 2007, I will call sentences like (30b) conditional conjunctions.

There are two intriguing connections with the free choice domain. First, free choice readings emerge when *or* disjoins tenseless propositions. That is, in *You may eat an apple or a pear*, *or* logically disjoins the tenseless propositions *you eat an apple* and *you eat a pear*. Likewise, the conditional conjunction requires that the first conjunct (typically) be tenseless (*strike this match*), and the second conjunct unmarked for tense with the paraphrastic futurate modal *will*.

Second, Russell argues that the first conjunct must in fact be syntactically an imperative, though it does not have imperative force. Imperatives, of course, are often used to express permission or obligation (*Have an apple!*), and, just like the permission sentences we have explored above, (30b) is all about discussing options.

If the imperative form somehow serves in place of the logically required negation, and also signals that we are reasoning about a resource-sensitive domain, then we would be in the position to not only understand this otherwise utterly mysterious use of *and*, but to actually predict that it should be possible.

9. CONCLUSIONS

On the view presented here, understanding free choice implications hinges on recognizing that permission is a scarce resource, and so requires a resource-sensitive semantics. Following Lokhorst 1997, I propose Linear Logic as a way of tracking permission as a limited resource: not only what kind of permission has been granted, but how much. Then primary free choice implications (given *You may eat an apple or a pear*, infer *You may eat an apple* with$_{\&}$ *You may eat a pear*) follow merely from expressing permission using the independently-motivated Anderson/Kanger deontic reduction. Double prohibition (from *You may not eat an apple or a pear* infer *You may not eat an apple* and *You may not eat a pear*) follows from the same Linear Logic analysis supplemented with standard Gricean reasoning.

In addition, I have proposed that *or* is ambiguous, and has among its meanings both $\oplus$ and &. Note that the primary free choice implications follow from assuming only that *or* has among its interpretations additive disjunction ($\oplus$). Similarly, double prohibition also follows from assuming that *or* can express additive disjunction. Therefore the explanations for the primary free choice implications and for double prohibition do not depend on assuming that *or* is ambiguous.

But if we assume that *or* is ambiguous, we also get an explanation for why unembedded *or* can give rise to a free choice reading (*You may eat an apple or you may eat a pear*). The hypothesis that *or* is ambiguous also accounts for strengthening, which is valid assuming that & is pronounced in English as *or*.

Linear Logic is one of the better known resource-sensitive logics; other logics may be worth considering instead. Similarly, the Anderson/Kanger deontic reduction strategy was adopted in part for ease of exposition, and work remains to integrate the account here within a more general framework of modality in natural language. But the point of this paper is not to advocate either Linear Logic specifically or the deontic reduction. Rather, my main goal is simply to suggest that we may be able to gain new and valuable insights into long-standing puzzles in natural language semantics if we allow ourselves to consider richer logical approaches than standard classical logic.

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